Mid-Term Exam

ChEn 693R: Dynamic Optimization

Feb 16-19, 2016

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Open Book, Notes, Homework, Internet

Time Limit: 6 hours (not necessarily consecutive)

Time(s) started: ___________________________________

Time(s) stopped: __________________________________

Cumulative time:__________________________________

Name__________________________________

1. (35 pts) Orthogonal Collocation on Finite Elements

Objective: Solve a set of differential equations with orthogonal collocation on finite elements.

Solve the following 2 coupled differential equations from time starting at 0 until a final time (t_f) of 10.0. Solve the system of equations with orthogonal collocation on finite elements with 4 nodes (time points) at $t = [0 \quad 5-\sqrt{5} \quad 5+\sqrt{5} \quad 10]$ for discretization points.

$$\frac{dx_1(t)}{dt} = u(t)$$

$$\frac{dx_2(t)}{dt} = x_1^2(t) - u(t)$$

$$x(0) = [-0.5, 0]$$

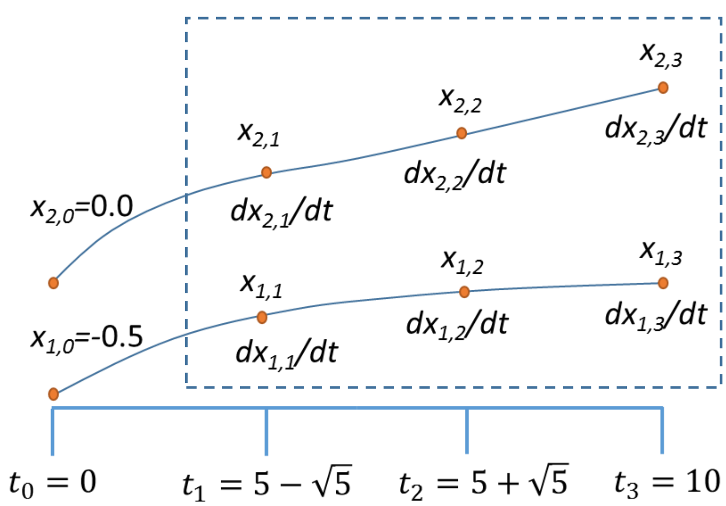
The initial conditions for x_1 and x_2 are -0.5 and 0, respectively. The value of the input, $u(t)$ is 0.5. The following is available for approximation of the derivative values at $x_{i,1}$, $x_{i,2}$, or $x_{i,3}$, where i is either variable 1 or 2 (see link for additional details on orthogonal collocation):

$$t_f \begin{bmatrix} 0.436 & -0.281 & 0.121 \\ 0.614 & 0.064 & 0.046 \\ 0.603 & 0.230 & 0.167 \end{bmatrix} \begin{bmatrix} \frac{dx_{i,1}}{dt} \\ \frac{dx_{i,2}}{dt} \\ \frac{dx_{i,3}}{dt} \end{bmatrix} = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \end{bmatrix} - \begin{bmatrix} x_{i,0} \\ x_{i,0} \\ x_{i,0} \end{bmatrix}$$

Report the solution of x_1 and $x_2$and derivative values ($\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$) at $t_1 = 5-\sqrt{5}$, $t_2 = 5+\sqrt{5}$, and $t_3 = 10$. Show work such as the system of 12 variables and 12 equations for full credit.

$x_{i,j}$ is *i*=variable index {1 or 2} and *j*=time index {0,1,2,3}. Solve for the following:

$x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, \frac{dx_{1,1}}{dt}, \frac{dx_{1,2}}{dt}, \frac{dx_{1,3}}{dt}, \frac{dx_{2,1}}{dt}, \frac{dx_{2,2}}{dt}, \frac{dx_{2,3}}{dt}$



2. (25 pts) Dynamic Optimization

Objective: Solve the following dynamic optimization problem.

$$\min_u \frac{1}{2}\int_0^{t_f} x_1^2(t)dt$$

$$s.t. \quad \frac{dx_1(t)}{dt} = u(t) + x_2(t)$$

$$\frac{dx_2(t)}{dt} = -u(t)$$

$$x(0) = [0.5, 0]$$

$$x(t_f) = [0, 0]$$

$$t_f = 1.5$$

$$-1 \leq u(t) \leq 1$$

Report the optimal objective value and display a plot of relevant variables (x_1, x_2, u). Note that this problem has both initial conditions for x_1 and x_2 as well as final conditions that must be satisfied. It is the integral of the objective, not just the final value that must be minimized by adjusting the value of u. Ensure that any solution has sufficient grid points to accurately represent the system dynamics and objective.

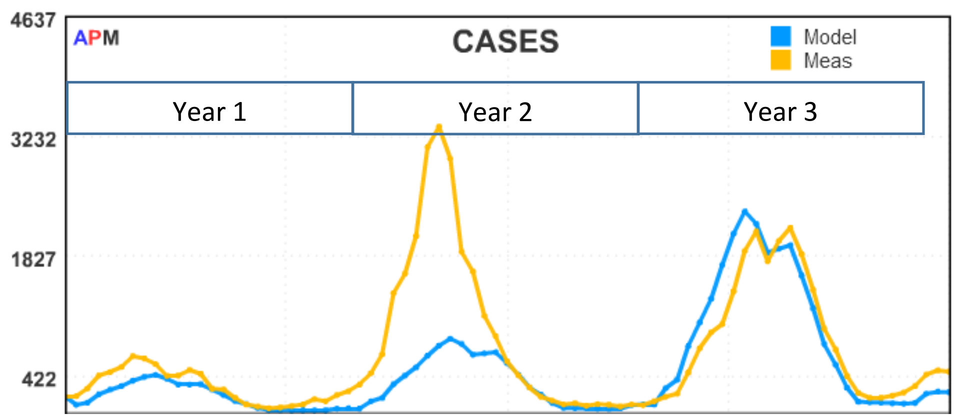
3. (40 pts) Dynamic Estimation and Optimization

Measles (sometimes known as English Measles) is spread through respiration (contact with fluids from an infected person's nose and mouth, either directly or through aerosol transmission), and is highly contagious—90% of people without immunity sharing living space with an infected person will catch it. The infection has an average incubation period of 14 days (range 6–19 days) and infectivity lasts from 2–4 days prior, until 2–5 days following the onset of the rash (i.e. 4–9 days infectivity in total).

Understanding the spread of the measles virus from historical data of major metropolitan areas helps researchers understand the fundamentals of how diseases spread through a population. This may help guide policy for public meeting bans, travel restrictions, and other measures intended to slow disease spread until a suitable vaccine can be developed.

Part A) Using the files at http://apmonitor.com/wiki/index.php/Apps/MeaslesVirus, estimate the bi-weekly transmission factors (β) and the recovery rate (γ), respectively denoted as *beta[i]* and *gamma* in the model. Note that the files are configured to solve the problem using a squared error objective. Below is the solution estimating *beta[1:26]* but not *gamma*. This is the result from the download if you select the plot for "cases" from the web-interface:

http://apmonitor.com/wiki/uploads/Apps/infectious_disease_estimation.zip



There are approximately 26 biweekly periods in the year, corresponding to the 26 beta values. The beta value from the 5th period of the year (beta[5]) is applied to each of 3 years. This beta value represents how much contact there is in the population. For example, during school sessions the contact term is higher than during vacation periods. The repeatable school sessions lead to periodic periods of increased infection.

Part B) Compare the squared error objective to the ℓ_1-norm objective solution. Estimate the 26 bi-weekly contact parameters (*beta[1:26]*) and the recovery rate (*gamma*).

Part C) Add a term to the population balance that includes a vaccine to reduce the susceptible population (*S*). One dose of vaccine reduces the susceptible population by one and the time units are one biweek per time step (see measles_biweek.csv). Write the modified population balance equation for the susceptible population (*S*) and add an equation for the vaccination supply (*V*) as a function of the vaccination rate (*Vr*). The available vaccine supply (*V*) is 200,000 (initial condition).

Original: $\frac{dS}{dt} = -rate + \mu N$

Modified: $\frac{dS}{dt} =$

Vaccine supply: $\frac{dV}{dt} =$

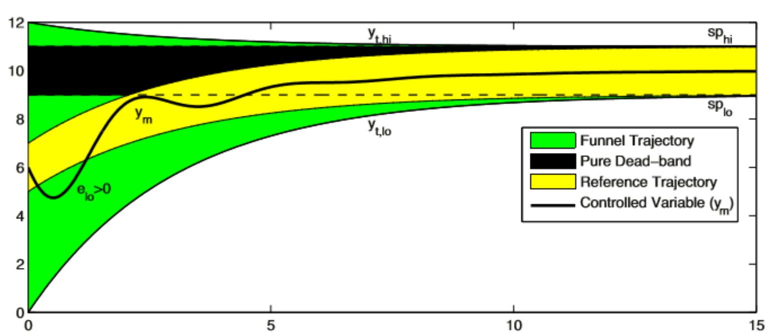
Part D) The vaccine has been developed but is in limited supply (see initial condition above). Determine the optimal dosing rate to prevent outbreaks over 3 years. Due to clinic availability, only 5,000 vaccines can be administered each week or 10,000 per biweek (2 week period). Design an optimizer that manipulates the dosing rate (*Vr*) to keep the number of reported infection cases always under 50. Use the best values of the parameters (*beta[1:26]* and *gamma*) that you determined from Part *B*. Show when the vaccine is applied by the optimizer and why is it a reasonable solution.

Hints for Part D:

1. The APMonitor plots have time scales of "years". It is actually "biweeks" for the time units of the time steps in the horizon.
2. The *biweek[i]* parameters in the model are simply to assign the correct value of *beta[i]* to the correct time step in the horizon.

time	cases	biweek[1]	biweek[2]	biweek[3]	biweek[4]	biweek[5]	biweek[6]
0	180	1	0	0	0	0	0
1	180	1	0	0	0	0	0
2	271	0	1	0	0	0	0
3	423	0	0	1	0	0	0
4	465	0	0	0	1	0	0
5	523	0	0	0	0	1	0
6	649	0	0	0	0	0	1

3. Add a parameter (*Vr*), variable (*V*), modify an equation ($\frac{dS}{dt} = -rate + \mu N$), and add an equation ($\frac{dV}{dt}$ =?) to the model. Set *Vr=0* initially to replicate the estimation solution with no vaccination.
4. Solve the estimation (part B) and control problem (part D) in separate applications. Start with the solution from part B and then add the control problem configuration to use the estimates of *beta* and *gamma*. There are beta (26 biweekly values) and gamma (1 value) that need to be transferred.
5. Below is sample configuration of a dead-band for a controlled variable "*x*" between an upper limit of 11 and a lower limit of 9.
   - apm_option(s,a,'x.tr_init',0)
   - apm_option(s,a,'x.sphi',11)
   - apm_option(s,a,'x.splo',9)



References

Daniel P. Word, James K. Young, Derek Cummings, Carl D. Laird, Estimation of seasonal transmission parameters in childhood infectious disease using a stochastic continuous time model, In: S. Pierucci and G. Buzzi Ferraris, Editor(s), Computer Aided Chemical Engineering, Elsevier, 2010, Volume 28, Pages 229-234, ISSN 1570-7946, ISBN 9780444535696, http://dx.doi.org/10.1016/S1570-7946(10)28039-2.