Chemical Engineering 436
Laplace Transforms (2)

Objective:
One step in the partial fraction expansion is to factor the denominator D(s) of the equation. We want to be able to tell about the behavior of y (transform = Y(s)) in the time domain from the factored expression prior to expansion.

Complex Factors:
In this class we will consider the situation where D(s) has complex roots. In other words, the factored polynomial yields terms of the form $s^2 + d_1 s + d_o$ where $d_1^2/4 < d_o$ (Why does this yield complex roots?). Note that complex roots come in pairs since we are dealing with a real system.

Example: The polynomial $s^2 + 4s + 5$ has complex roots which can be found with use of the quadratic formula. (Alternately, Mathcad or some calculators will due this for you).

$$\text{Roots:} \quad \frac{-4 \pm \sqrt{4^2 - (4)(5)}}{2} = -2 \pm j$$

Therefore, $s^2 + 4s + 5$ has the factors (s+2+j)(s+2-j) (Note that $j = \sqrt{-1}$)

Important: the roots are (-2+j) and (-2-j), but the factors are (s+2-j)(s+2+j).

Key Issues with regards to complex factors
1) Complex roots indicate oscillatory behavior.
2) If the sign of the real part of the complex roots is negative, convergence is expected. Conversely, if the real part is positive, it will diverge.
3) Algebra needed to invert transforms with complex roots is messy but doable.
4) We don't need to invert the transform to tell whether it will converge or diverge, or whether or not it will oscillate.

Practice:
Will y(t) converge or diverge? Is y(t) smooth or oscillatory?

$$Y(s) = \frac{s+2}{s(s^2 + 4s + 13)}$$

Inverting Transforms with Complex Roots in the Denominator

There are at least two different ways to proceed as described in your text on p. 63-65.

1) Use of complex numbers and Euler's identity
2) Expansion without using complex numbers, followed by completing the square to invert the transform.

We will examine the second of these in detail in order to show the connection between the complex factors and oscillation in the time domain. Example:

$$\frac{s + 2}{s(s^2 + 4s + 5)}$$

1. Perform the partial fraction expansion (leaving the quadratic expression with complex roots unfactored)

$$Y(s) = \frac{s+2}{s(s^2+4s+5)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2+4s+5}$$

Note that two coefficients are needed when there is a second order polynomial in the denominator, and that one of them is multiplied by s.

We can use the Heaviside Expansion to find α_1. $\alpha_1 = \frac{2}{5}$ (verify this)

To solve for the coefficients of the quadratic term, we need to clear the denominator and group "like" powers of s like we did for repeated factors.

$$s+2 = \alpha_1(s^2+4s+5) + s(\alpha_2 s + \alpha_3) = (\alpha_1+\alpha_2)s^2 + (4\alpha_1+\alpha_3)s + \alpha_1 5$$

$$\alpha_2 = -\alpha_1 = -\frac{2}{5}$$

$$\alpha_3 = 1 - 4\alpha_1 = -\frac{3}{5}$$

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{s^2+4s+5}$$

2. Complete the square on the denominator of quadratic terms in order to put them into the proper form for inversion

$$s^2 + 4s + 5 = (s+b)^2 + w^2$$

To do this, b = (coefficient in front of s term)/2 = 4/2 = 2.

Once b is known, we can back solve for w. $b^2 + w^2 = 5$. Therefore, $w = \sqrt{5-2^2} = 1$

As a double check: $(s + 2)^2 + 1 = (s^2 + 4s + 4) + 1 = s^2 + 4s + 5$

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{(s+2)^2+1}$$

3. Rearrange into the form listed in the transform table and invert

Because there are complex roots, we know that we have an oscillating solution and need either a sine or cosine term, or both. Furthermore, the "s" in the numerator indicates that we have a cosine term (this does not exclude a sine). For the cosine, we need an s+2 in the numerator. Let's set up the cosine term and put whatever is left in the sine term.

$$\frac{-2}{5}\left[\frac{(s+2)}{(s+2)^2+1}\right]$$

The -2/5 was factored out to give the "s" in the numerator. This expression plus a similar one for a sine term must add to yield the second term in the expression for Y(s) above.

$$\frac{-\frac{2}{5}s-\frac{3}{5}}{(s+2)^2+1}=\frac{-\frac{2}{5}(s+2)+\frac{1}{5}}{(s+2)^2+1}=-{}^{2}\!/\!_{5}\left[\frac{(s+2)}{(s+2)^2+1}\right]+\frac{1}{5}\left[\frac{1}{(s+2)^2+1}\right]$$

The resulting expression for Y(s) is now:

$$Y(s)=\frac{2}{5s}-{}^{2}\!/\!_{5}\left[\frac{(s+2)}{(s+2)^2+1}\right]+{}^{1}\!/\!_{5}\left[\frac{1}{(s+2)^2+1}\right]$$

This is easily inverted to yield:

$$y(t)=\frac{2}{5}-\frac{2}{5}e^{-2t}\cos t+\frac{1}{5}e^{-2t}\sin t$$

Denominators with complex roots will always lead to oscillations (sine and cosine terms).
Will the above expression for y(t) converge or diverge? Why?

Practice

Please use the above method to invert the following expression: $Y(s)=\dfrac{1}{s^2-4s+13}$

Note that no partial fraction expansion is needed- only one "term" in the denominator.

Let's suppose that you plug the denominator D(s) of a messy looking transform into Mathcad to find the roots. Mathcad provides the following output (using the polyroots function):

$$\begin{bmatrix} 2+6i \\ 2-6i \\ -1 \\ -3 \\ -2 \end{bmatrix}$$

Assuming that N(s) = 1, write out the factored form of Y(s) = N(s)/D(s). Is y(t) converging or diverging?

Other important topics: (you should understand these and be able to use them)

- Transform of an integral (p. 61) $\pounds\left\{\int_0^t f\left(t^*\right)dt^*\right\} = \frac{1}{s}F(s)$
- Initial value theorem (p. 60) find the initial value of $\frac{(s+2)}{(s+3)(s+4)}$
- Final value theorem (p. 59) find the final value of $\frac{(s+6)}{(s+1)(s+2)}$
- Real translation theorem (p. 62) (i.e., time delays)

Note: You must first determine whether the transform will lead to convergent or divergent behavior in the time domain before attempting to apply the final value theorem.