

Chemical Engineering 436 Laplace Transforms (1)

Why Laplace Transforms??

- 1) Converts differential equations to algebraic equations- facilitates combination of multiple components in a system to get the total dynamic behavior (through addition & multiplication)
- 2) Can gain insight from the solution in the transform domain ("s")- inversion of transform not necessarily required
- 3) Allows development of an analytical model which permits use of a discontinuous (piecewise continuous) forcing function and the use of an integral term in the forcing function (important for control)

Definition of a Laplace Transform

$$F(s) = L[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$
$$L^{-1}[F(s)] = f(t)$$

Note that since the Laplace function is an integral, the Laplace of a sum is the sum of the Laplace of each term.

$$L[a \cdot f(t) + b \cdot g(t)] = a \cdot L[f(t)] + b \cdot L[g(t)]$$

Table of Transforms

- 1) Table 3.1 (see pp. 42-43 of text)
- 2) **Practice:** Use the table to find the Laplace transform of the following:
 - a. 1000 S(t) (Step function with a magnitude of 1000)
 - b. $5e^{-6t} + \sin 4t + 5$
 - c. $\frac{d^3 y}{dt^3}$ where $\left(\frac{d^2 y}{dt^2}\right)_{t=0} = 0$, $\left(\frac{dy}{dt}\right)_{t=0} = 2$, $y(0) = 3$

Solution of Differential Equations

- 1) Transform both sides of the differential equation to yield an algebraic equation in s
- 2) Substitute values for the initial conditions of derivatives as applicable
- 3) Solve the resulting equation for the output variable to yield the transform of the output variable on the right side of the equation
- 4) Invert the transform of the output variable to give the time domain solution

Example: Use Laplace transforms to solve the following equation

$$\frac{dy}{dt} + 3y = e^{-2t}, \quad y(0) = 2$$

Characteristic Polynomial

The characteristic polynomial which corresponds to a linear differential equation of the form:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

is defined as:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

Whenever you take the transform of a linear, constant coefficient ODE of the form shown above, you will end up with an expression on the left-hand side of the equation where $Y(s)$, the transform of $y(t)$, is multiplied by the characteristic polynomial. In other words,

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0)Y(s) = RHS$$

where the RHS is a function of s . Solving for $Y(s)$ will yield an expression with the characteristic polynomial in the denominator:

$$Y(s) = \frac{RHS}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

Partial Fraction Expansion

Used to permit inversion of transforms which contain higher-order polynomials of s than found in the transform table.

- 1) Factor the denominator which consists of the characteristic polynomial from the differential equation and other terms contributed by inputs (Why is this the case?). Usually the contributions from the inputs are not complex. Therefore, the most difficult part of this task is factoring the characteristic polynomial. Equation solvers such as Mathcad (or your calculators?) can be used to find all of the roots of the characteristic polynomial. You will want to use the symbolic capability to get all roots at once. Note that you may have complex roots- we will talk about these next time.

For example, let's suppose that we have an expression in the Laplace domain which we want to invert:

$$F(s) = \frac{12s^2 + 22s + 6}{s(s^2 + 3s + 2)}$$

where $1/s$ is the "input" contribution and $s^2 + 3s + 2$ is the characteristic polynomial (what was the differential equation which yielded this polynomial?). Let $N(s)$ represent the numerator and $D(s)$ represent the denominator. In this case, the denominator can be easily factored to yield:

$$F(s) = \frac{12s^2 + 22s + 6}{s(s+1)(s+2)}$$

- 2) Express the function of s with the factored denominator as a sum of partial fractions as follows (eq. 3-46):

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^n (s + b_i)} = \sum_{i=1}^n \frac{\alpha_i}{s + b_i}$$

For example:

$$\frac{12s^2 + 22s + 6}{s(s+1)(s+2)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2}$$

3) Calculate values for the α 's from the Heaviside expansion (see p. 59, Method 3) where the $s + b_i$ terms represent the factored terms in the denominator ($s, s + 1, s + 2$ for the example above). $-b_i$ is the value of s that makes $s + b_i$ equal to zero ($0, -1, -2$ for the above example). The expansion can be written as follows:

$$\alpha_i = (s + b_i) \left. \frac{N(s)}{D(s)} \right|_{s=-b_i} \quad \text{For example:}$$

$$\alpha_1 = (s) \left[\frac{12s^2 + 22s + 6}{s(s+1)(s+2)} \right]_{s=0} = \frac{6}{2} = 3$$

Note that Heaviside is uppercase as it refers to a name (Oliver Heaviside). I would expect you to be able to perform partial fraction expansion on an exam, as well as some simple factoring.

Practice: Determine α_2 and α_3 in the above example and then use partial fractions to invert the transform.

Repeated Factors

The Heaviside expansion cannot be used to obtain all of the coefficients for partial fraction expansion when repeated factors (e.g., s^2 or $(s+1)^3$, etc) are present. Page 47 of your text provides an example of how these repeated factors may be handled. Personally, I prefer the method mentioned on p. 47 at the bottom of Example 3.1 where you multiply through by the denominator of the expression you are trying to expand and then group "like powers" of s to yield a set of equations which can be solved for the α 's. You may want to refer to your Math 113 book for additional information if needed.

Example: Repeated Factors

$$F(s) = \frac{s+1}{(s+3)^2} = \frac{\alpha_1}{s+3} + \frac{\alpha_2}{(s+3)^2}$$

Multiplying through by $(s+3)^2$ gives:

$$s+1 = \alpha_1(s+3) + \alpha_2 = \alpha_1 s + (3\alpha_1 + \alpha_2)$$

Consequently, looking at “like” powers of s : $\alpha_1 = 1$ and $\alpha_2 = -2$

Practice: Invert $F(s)$ in the above example to get the time domain solution.

As a postscript, Mathcad can be used to do the partial fraction expansion. Once the partial fraction expansion has been performed, you can invert the transform term by term using Table 3.1. As if that weren't enough, Mathcad will also invert the transform for you directly. Simply type in $F(s)$, put the cursor somewhere on the expression, and choose “Inverse Laplace Transform” under Transforms in the Symbolic menu (Version 6, Professional). You are welcome to use Mathcad to check your work. **However, the homework should be done by hand in order to receive credit.**

Additional notes: (all the relevant Laplace equations are not in Table 3.1)

1. **Final value theorem** (Eq. 3-91) $y(\infty) = \lim_{s \rightarrow 0} [sY(s)]$

2. **Initial value theorem** (Eq. 3-95) $y(0) = \lim_{s \rightarrow \infty} [sY(s)]$

3. **Time delay** (Real Translation Theorem, Eq. 3-104)

$$G(s) = L\{f(t-t_0)S(t-t_0)\} = e^{-st_0} F(s)$$

- In time domain, replace all t 's with $(t-t_0)$
- In Laplace domain, put e^{-st_0} in front of Laplace domain term

Practice: Write the Laplace form of a function that does the doublet test,

- (a) changing at $t=0$ to a value of 2,
- (b) changing to a value of -2 at $t = 3$ min, and
- (c) changing to a value of 0 at $t = 6$ min.

Practice: Write the time domain form of the following Laplace function and sketch it:

$$e^{-3s} \frac{3}{s^2} - e^{-6s} \frac{6}{s^2} + e^{-9s} \frac{3}{s^2}$$

Practice: Determine the initial and final value of the following function:

$$F(s) = \frac{12s^2 + 22s + 6}{s(s+1)(s+2)}$$