Chapter 3

Class 16
Laplace Transforms

Laplace Transforms

- Important analytical method for solving *linear* ordinary differential equations.

    - Application to nonlinear ODEs? Must linearize first.

- Laplace transforms play a key role in important process control concepts and techniques.

    - Examples:

        - Transfer functions
        - Frequency response
        - Control system design
        - Stability analysis

Definition

The Laplace transform of a function, $f(t)$, is defined as

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^{\infty} f\left(t\right) e^{-st} dt \qquad (3\text{-}1)$$

where $F(s)$ is the symbol for the Laplace transform, $\mathcal{L}$ is the Laplace transform operator, and $f(t)$ is some function of time, t.

Note: The $\mathcal{L}$ operator transforms a time domain function $f(t)$ into an s domain function, $F(s)$.

Inverse Laplace Transform, L^{-1}:

By definition, the inverse Laplace transform operator, L^{-1}, converts an *s*-domain function back to the corresponding time domain function:

$$f(t) = L^{-1}\left[F(s)\right]$$

Important Properties:

Both L and L^{-1} are *linear operators*. Thus,

$$\begin{aligned} L\left[ax(t)+by(t)\right] &= aL\left[x(t)\right]+bL\left[y(t)\right] \\ &= aX(s)+bY(s) \qquad (3\text{-}3) \end{aligned}$$

where:

- $x(t)$ and $y(t)$ are arbitrary functions
- a and b are constants
- $X(s) = L[x(t)] \ \ and \ \ Y(s) = L[y(t)]$

Similarly,

$$\mathsf{L}^{-1}\left[aX(s)+bY(s)\right] = ax(t)+by(t)$$

Laplace Transforms of Common Functions

1. Constant Function

Let $f(t) = a$ (a constant). Then from the definition of the Laplace transform in (3-1),

$$\mathcal{L}(a) = \int_0^{\infty} a e^{-st} dt = -\frac{a}{s} e^{-st} \Bigg|_0^{\infty} = 0 - \left(-\frac{a}{s}\right) = \boxed{\frac{a}{s}} \qquad (3\text{-}4)$$

2. Step Function

The unit step function is widely used in the analysis of process control problems. It is defined as:

$$S(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \qquad (3\text{-}5)$$

Because the step function is a special case of a "constant", it follows from (3-4) that

$$\mathcal{L}\left[S(t)\right] = \frac{1}{s} \qquad (3\text{-}6)$$

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3. Derivatives

This is a very important transform because derivatives appear in the ODEs we wish to solve. In the text (p.41), it is shown that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \qquad (3\text{-}9)$$

initial condition at $t = 0$

Similarly, for higher order derivatives:

First derivative

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \\ - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \qquad (3\text{-}14)$$

where:

- n is an arbitrary positive integer

- $f^{(k)}(0) = \left.\dfrac{d^k f}{dt^k}\right|_{t=0}$

Special Case: All Initial Conditions are Zero

Suppose $f(0) = f^{(1)}(0) = \ldots = f^{(n-1)}(0)$. Then

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s)$$

In process control problems, we usually assume zero initial conditions. *Reason:* This corresponds to the nominal steady state when "deviation variables" are used, as shown in Ch. 4.

4. Exponential Functions

Consider $f(t) = e^{-bt}$ where $b > 0$. Then,

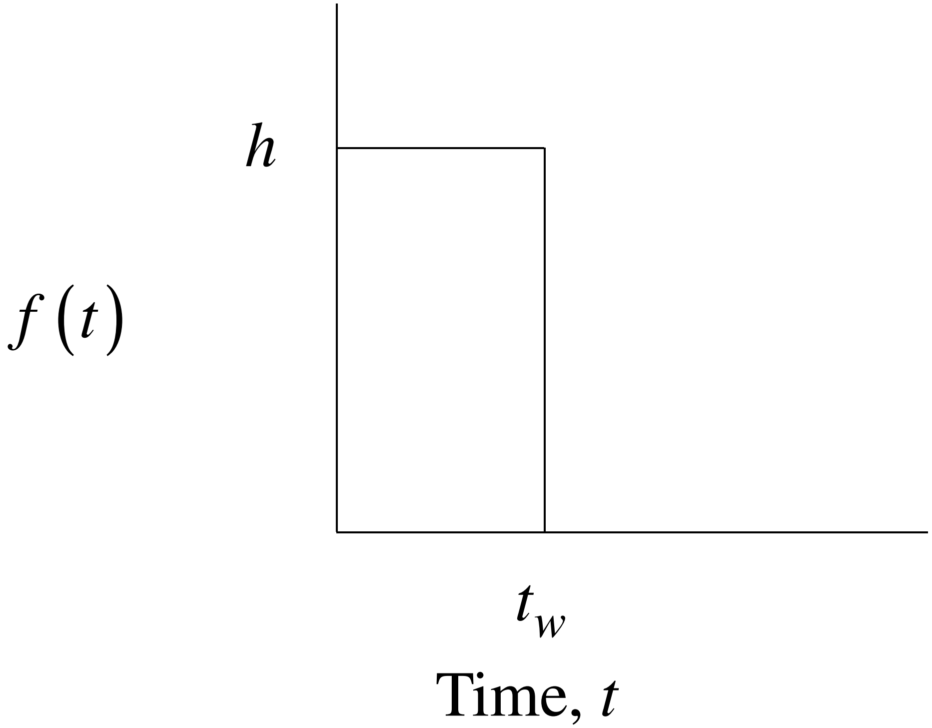
$$\mathcal{L}\left[e^{-bt}\right] = \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt$$

$$= \frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty = \boxed{\frac{1}{s+b}} \qquad (3\text{-}16)$$

5. Rectangular Pulse Function

It is defined by:

$$f(t) = \begin{cases} 0 & \text{for } \ t < 0 \\ h & \text{for } \ 0 \le t < t_w \\ 0 & \text{for } \ t \ge t_w \end{cases} \qquad (3\text{-}20)$$



The Laplace transform of the rectangular pulse is given by

$$F(s) = \frac{h}{s}\left(1 - e^{-t_w s}\right) \qquad (3\text{-}22)$$

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6. Impulse Function (or Dirac Delta Function)

The impulse function is obtained by taking the limit of the rectangular pulse as its width, t_w, goes to zero but holding the area under the pulse constant at one. (i.e., let $h = \frac{1}{t_w}$)

Let, $\delta(t)$ = impulse function

Then, $\mathcal{L}\left[\delta(t)\right] = 1$

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Table 3.1. Laplace Transforms

See page 42 of the text.

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$		$F(s)$
1. $\delta(t)$	(unit impulse)	1
2. $S(t)$	(unit step)	$\frac{1}{s}$
3. t	(ramp)	$\frac{1}{s^2}$
4. t^{n-1}		$\frac{(n-1)!}{s^n}$
5. e^{-bt}		$\frac{1}{s+b}$
6. $\frac{1}{\tau}e^{-t/\tau}$		$\frac{1}{\tau s+1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$	$(n>0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$		$\frac{1}{(\tau s+1)^n}$
9. $\frac{1}{b_1-b_2}(e^{-b_2t}-e^{-b_1t})$		$\frac{1}{(s+b_1)(s+b_2)}$
10. $\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$		$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11. $\frac{b_3-b_1}{b_2-b_1}e^{-b_1t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2t}$		$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$		$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13. $1-e^{-t/\tau}$		$\frac{1}{s(\tau s+1)}$

Laplace table (cont.)

14. $\sin \omega t$		$\dfrac{\omega}{s^2+\omega^2}$
15. $\cos \omega t$		$\dfrac{s}{s^2+\omega^2}$
16. $\sin(\omega t + \phi)$		$\dfrac{\omega \cos \phi + s \sin \phi}{s^2+\omega^2}$
17. $e^{-bt} \sin \omega t$	b, ω real	$\dfrac{\omega}{(s+b)^2+\omega^2}$
18. $e^{-bt} \cos \omega t$	b, ω real	$\dfrac{s+b}{(s+b)^2+\omega^2}$
19. $\dfrac{1}{\tau\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left(\sqrt{1-\zeta^2}\, t/\tau\right)$ $(0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20. $1 + \dfrac{1}{\tau_2 - \tau_1}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)$ $(\tau_1 \ne \tau_2)$		$\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left[\sqrt{1-\zeta^2}\, t/\tau + \psi\right]$ $\psi = \tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}\left[\cos\left(\sqrt{1-\zeta^2}\, t/\tau\right) + \dfrac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\, t/\tau\right)\right]$ $(0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \dfrac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \dfrac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \ne \tau_2)$		$\dfrac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\dfrac{df}{dt}$		$sF(s) - f(0)$
25. $\dfrac{d^n f}{dt^n}$		$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0) S(t - t_0)$		$e^{-t_0 s} F(s)$

Practice

a. 1000 **S**(t) (Step function with a magnitude of 1000)

$$\frac{1000}{s}$$

b. $5e^{-6t}$ + sin 4t + 5

$$\frac{5}{s+6}+\frac{4}{s^2+16}+\frac{5}{s}$$

c. $\dfrac{d^3y}{dt^3} \quad where \left(\dfrac{d^2y}{dt^2}\right)_{t=0}=0, \left(\dfrac{dy}{dt}\right)_{t=0}=2, \quad y(0)=3$

$$s^3F(s)-s^2(3)-s(2)-0=s^3F(s)-3s^2-s$$

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Solution of ODEs by Laplace Transforms

Procedure:

1. Take the L of both sides of the ODE.
2. Rearrange the resulting algebraic equation in the *s* domain to solve for the L of the output variable, e.g., $Y(s)$.
3. Perform a partial fraction expansion.
4. Use the L^{-1} to find $y(t)$ from the expression for $Y(s)$.

Practice

Solve the following equation:

$$\frac{dy}{dt} + 3y = e^{-2t} \qquad y(0) = 2$$

$$sY(s) - 2 + 3Y(s) = \frac{1}{s+2}$$

$$(s+3)Y(s) - 2 = \frac{1}{s+2}$$

$$y(t) = e^{-2t} + e^{-3t}$$

$$(s+3)Y(s) = 2 + \frac{1}{s+2} = \frac{2s+4+1}{s+2} = \frac{2s+5}{s+2}$$

$$Y(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{2(s+5/2)}{(s+2)(s+3)}$$

Use #11 in Table 3.1

$$y(t) = 2\left[\left(\frac{5/2-2}{3-2}\right)e^{-2t} + \left(\frac{5/2-2}{3-2}\right)e^{-3t}\right]$$

$$y(t) = e^{-2t} + e^{-3t}$$

Check Answer:

$$y(0) = 1 + 1 = 2$$

$$y'(t) = -2e^{-2t} - 3e^{-3t}$$

$$3y(t) = 3e^{-2t} + 3e^{-3t}$$

$$y'(t) + 3y(t) = e^{-2t}$$

Partial Fraction Expansions

Basic idea: Expand a complex expression for $Y(s)$ into simpler terms, each of which appears in the Laplace Transform table. Then you can take the L^{-1} of both sides of the equation to obtain $y(t)$.

Example:

$$Y(s) = \frac{s+5}{(s+1)(s+4)} \tag{3-41}$$

Perform a partial fraction expansion (PFE)

$$\frac{s+5}{(s+1)(s+4)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} \tag{3-42}$$

where coefficients α_1 and α_2 have to be determined.

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To find α_1*: Multiply both sides by s + 1 and let s = -1*

$$\therefore \quad \alpha_1 = \left.\frac{s+5}{s+4}\right|_{s=-1} = \frac{4}{3}$$

To find α_2*: Multiply both sides by s + 4 and let s = -4*

$$\therefore \quad \alpha_2 = \left.\frac{s+5}{s+1}\right|_{s=-4} = -\frac{1}{3}$$

A General PFE

Consider a general expression,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\overset{n}{\underset{i=1}{\pi}}(s+b_i)} \qquad \text{(3-46a)}$$

Here $D(s)$ is an n-th order polynomial with the roots $(s = -b_i)$ all being *real* numbers which are *distinct* so there are no repeated roots.

The PFE is:

$$Y(s) = \frac{N(s)}{\overset{n}{\underset{i=1}{\pi}}(s+b_i)} = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i} \qquad \text{(3-46b)}$$

Note: $D(s)$ is called the "characteristic polynomial".

Special Situations:

Two other types of situations commonly occur when $D(s)$ has:

i) Complex roots: e.g., $b_i = 3 \pm 4j \qquad \left(j = \sqrt{-1}\right)$

ii) Repeated roots (e.g., $b_1 = b_2 = -3$)

For these situations, the PFE has a different form. See SEM text (pp. 47-48) for details.

Partial Fraction Example

$$\frac{12s^2+22s+6}{s(s+1)(s+2)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} \quad \boxed{= \frac{3}{s} + \frac{4}{s+1} + \frac{5}{s+2}}$$

To get α_1, multiply both sides by s and set s = 0

$$\frac{s(12s^2+22s+6)}{s(s+1)(s+2)} = \frac{s\alpha_1}{s} + \frac{s\alpha_2}{s+1} + \frac{s\alpha_3}{s+2}$$

$$\frac{(12\cdot 0^2+22\cdot 0+6)}{(0+1)(0+2)} = \alpha_1 = 6/2 = 3$$

Now get α_2:

$$\frac{(12\cdot(-1)^2+22\cdot(-1)+6)}{(-1)((-1)+2)} = \alpha_2 = \frac{-4}{-1} = 4$$

Finally get α_3:

$$\frac{(12\cdot(-2)^2+22\cdot(-2)+6)}{(-2)((-2)+1)} = \alpha_3 = \frac{10}{2} = 5$$

So now solve for f(t):

$$\boxed{f(t) = 3 + 4e^{-t} + 5e^{-2t}}$$

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Repeated Factors

$$F(s) = \frac{s+1}{(s+3)^2} = \frac{\alpha_1}{s+3} + \frac{\alpha_2}{(s+3)^2}$$

How do you get α_1 and α_2?

Multiply out denominators and match “like” powers of s.

$$\frac{(s+1)(s+3)^2}{(s+3)^2} = \frac{\alpha_1 (s+3)^2}{s+3} + \frac{\alpha_2 (s+3)^2}{(s+3)^2}$$

$$(s+1) = \alpha_1 (s+3) + \alpha_2 = s(\alpha_1) + (3\alpha_1 + \alpha_2)$$

Therefore, $\alpha_1 = 1$, and $3\alpha_1 + \alpha_2 = 1$. This means that $\alpha_2 = -2$.

So $$F(s) = \frac{s+1}{(s+3)^2} = \frac{1}{s+3} + \frac{-2}{(s+3)^2}$$

Inverting $$f(t) = e^{-3t} - 2te^{-3t}$$

Additional Notes

1. Final value theorem (Eq. 3-81)

$$y(\infty) = \lim_{s \to 0} [s\,Y(s)]$$

2. Initial value theorem (Eq. 3-82)

$$y(0) = \lim_{s \to \infty} [s\,Y(s)]$$

3. Time delay
(Real Translation Theorem, Eq. 3-96)

$$G(s) = L\{f(t - t_0)S(t - t_0)\} = e^{-s\,t_0} F(s)$$

More Practice

Practice: Write the Laplace form of a function that does the doublet test,

(a) changing at t=0 to a value of 2,

(b) changing to a value of -2 at t = 3 min, and

(c) changing to a value of 0 at t = 6 min.

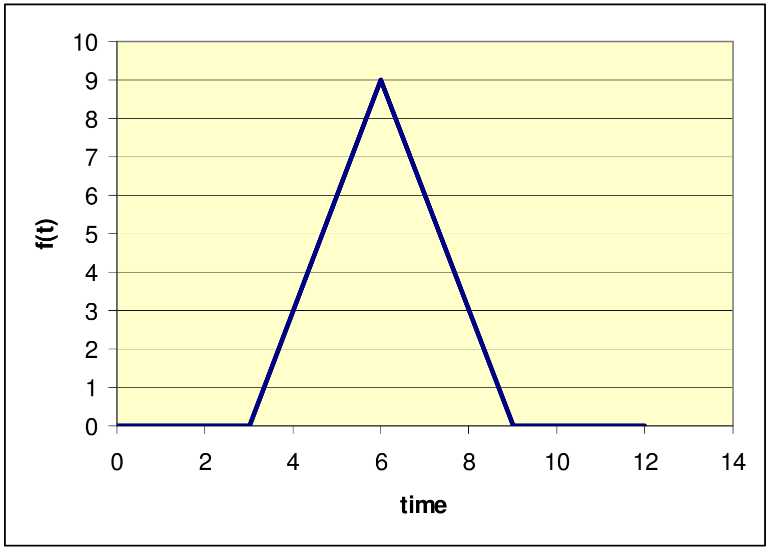
$$\frac{2}{s} + e^{-3s}\left(\frac{-4}{s}\right) + e^{-6s}\left(\frac{2}{s}\right)$$

More Practice

Write the time domain form of the following Laplace function and sketch it:

$$e^{-3s}\frac{3}{s^2} - e^{-6s}\frac{6}{s^2} + e^{-9s}\frac{3}{s^2}$$

$$3(t-3)[S(t-3)] - 6(t-6)[S(t-6)] + 3(t-9)[S(t-9)]$$



More Practice

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Determine the final value of the following function:

$$F(s) = \frac{12s^2 + 22s + 6}{s(s+1)(s+2)}$$

$$F(s=0) = \frac{s(12s^2 + 22s + 6)}{s(s+1)(s+2)} = \frac{(6)}{(1)(2)} = 3$$

Extra

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Example 3.1

Solve the ODE,

$$5\frac{dy}{dt}+4y=2 \qquad y(0)=1 \qquad (3\text{-}26)$$

First, take $\mathcal{L}$ of both sides of (3-26),

$$5\left(sY(s)-1\right)+4Y(s)=\frac{2}{s}$$

Rearrange,

$$Y(s)=\frac{5s+2}{s(5s+4)} \qquad (3\text{-}34)$$

Take $\mathcal{L}^{-1}$,

$$y(t)=\mathcal{L}^{-1}\left[\frac{5s+2}{s(5s+4)}\right]$$

From Table 3.1,

How do you get (3-37)?

$$\boxed{y(t)=0.5+0.5e^{-0.8t}} \qquad (3\text{-}37)$$

Partial Fraction Expansion

$$y(t) = \mathrm{L}^{-1}\left[\frac{5s+2}{s(5s+4)}\right]$$

$$\frac{5s+2}{s(5s+4)} = \frac{s+\frac{2}{5}}{s\left(s+\frac{4}{5}\right)} = \frac{s+0.4}{s(s+0.8)}$$

$$\frac{s+0.4}{s(s+0.8)} = \frac{s}{s(s+0.8)} + \frac{0.4}{s(s+0.8)} = \frac{1}{(s+0.8)} + \frac{0.4}{s(s+0.8)}$$

$$L^{-1}\left\{\frac{1}{(s+0.8)} + \frac{0.4}{s(s+0.8)}\right\} = e^{-0.8t} + 0.4\left[\frac{1}{-0.8}\left(e^{-0.8t}-1\right)\right]$$

$$= e^{-0.8t} - 0.5\left[\left(e^{-0.8t}-1\right)\right] = 0.5 + 0.5e^{-0.8t}$$

(#9 in table with b1 = 0)

Example 3.2 (continued)

Recall that the ODE, $\dddot{y} + +6\ddot{y} + 11\dot{y} + 6y = 1$, with zero initial conditions resulted in the expression

$$Y(s) = \frac{1}{s\left(s^3 + 6s^2 + 11s + 6\right)} \tag{3-40}$$

The denominator can be factored as

$$s\left(s^3 + 6s^2 + 11s + 6\right) = s(s+1)(s+2)(s+3) \tag{3-50}$$

Note: Normally, numerical techniques are required in order to calculate the roots.

The PFE for (3-40) is

$$Y(s) = \frac{1}{s(s+1)(s+2)(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3} \tag{3-51}$$

Solve for coefficients to get

$$\alpha_1 = \frac{1}{6}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = -\frac{1}{6}$$

(For example, find α, by multiplying both sides by s and then setting $s = 0$.)

Substitute numerical values into (3-51):

$$Y(s) = \frac{1/6}{s} - \frac{1/2}{s+1} + \frac{1/2}{s+2} + \frac{1/6}{s+3}$$

Take $\mathcal{L}^{-1}$ of both sides:

$$\mathcal{L}^{-1}\left[Y(s)\right] = \mathcal{L}^{-1}\left[\frac{1/6}{s}\right] - \mathcal{L}^{-1}\left[\frac{1/2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1/2}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1/6}{s+3}\right]$$

From Table 3.1,

$$y(t) = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t} \qquad (3\text{-}52)$$

Important Properties of Laplace Transforms

1. Final Value Theorem

It can be used to find the steady-state value of a closed loop system (providing that a steady-state value exists.

Statement of FVT:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[sY(s) \right]$$

providing that the limit exists (is finite) for all $\mathrm{Re}(s) \geq 0$, where Re (s) denotes the real part of complex variable, s.

Example:

Suppose,

$$Y(s) = \frac{5s+2}{s(5s+4)} \tag{3-34}$$

Then,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\frac{5s+2}{5s+4}\right] = 0.5$$