INTERACTIVE CONTROL ENGINEERING
Principles and Applications

Volume 19 Control of Biological Systems and Human Performance

CHAPTER 6

Treadmill Pacing Simulator for Constraint Selection and Fixed-Horizon Optimization

Pages 137–168

Edited by John D. Hedengren

How to cite this chapter. "Treadmill Pacing Simulator for Constraint Selection and Fixed-Horizon Optimization," in *Control of Biological Systems and Human Performance*, John D. Hedengren, Ed., vol. 19 of *Interactive Control Engineering: Principles and Applications*. Advanced Process Solutions, LLC, 2026, ch. 6, pp. 137–168.

The page numbers in this file are those of the printed volume.

The app for this chapter
https://apmonitor.com/apps/treadmill-runner/

The volume online: contents, other chapters and citation records
https://apmonitor.com/apps/books/v19-biological-human/#treadmill-runner

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CHAPTER 6

Treadmill Pacing Simulator for Constraint Selection and Fixed-Horizon Optimization



The controller moves 1 motor that sets the belt speed

The controller reads 1 heart rate, from a chest strap and its receiver 2 blood lactate, which the chapter estimates

What disturbs it 1 tilt of the deck

One motor sets the belt speed for two readings of the runner, heart rate and blood lactate, while the tilt of the deck changes what every speed costs. Illustration, not to scale.

IN PLAIN TERMS **Pacing a Runner on a Treadmill**

The main idea. A treadmill has one control, the speed of its belt. A controller sets that speed so that a runner covers as much ground as possible in one hour without passing safe limits of the body.

The challenge. Heart rate, body heat, stored fuel and blood lactate, a sign of hard effort, all climb with speed at different rates, and the first to reach its limit must set the speed. Extra effort left unused when the hour ends is distance thrown away.

Why it matters. The same problem appears wherever one control has to respect several limits before a fixed deadline.

THREE WAYS TO READ THIS CHAPTER

Explore	12 min	The first page, the fact box of Section 6.1, the history boxes of Sections 6.2 and 6.3, the try-it boxes of Sections 6.3 and 6.7, the fact box of Section 6.7 and the key points
Study	60 min	The whole chapter with its pause points and the twelve exercises
Practice	20 min	The first page, Section 6.7, Section 6.8, Section 6.9 and the open challenge

The control question Belt speed is the only actuator of a treadmill, while blood lactate, heart rate, core temperature and fuel limit how fast it may run and answer on different time scales. How should one actuator be shared among those limits, and what is a finite store worth when the hour has a fixed end?

Learning objectives

- Compute the static gain from belt speed to heart rate, and explain why gains carried as a fraction of threshold pace serve every runner.
- Predict the time a runner lasts above critical power from the reserve and the excess power.
- Explain how a low selector with an external-reset band shares one actuator, and why tracking the losing loop to the applied output deadlocks.
- Measure what incline feedforward and an aggressive heart-rate tuning do to distance, command travel and lactate.
- Divide the predictive controller's advantage into a tuning choice and an architectural part, and judge what its terminal value is worth.

Prerequisites Sections 1.6.2 (velocity form), 1.4.4 (scheduling on a known individual number), 1.4.6 (overrides and stores) and 1.6.4 (scenarios and data). Section 1.6.3 is needed only for the predictive controller.

The app https://apmonitor.com/apps/treadmill-runner/

See also Volume 4, Chapter 4 and Volume 4, Chapter 3 for one actuator shared among limits through a selector whose losing loops track the applied output. Volume 9, Chapter 12 for slack spent against a fixed deadline, and Chapter 7 for the same athlete model behind an actuator whose authority is a store.

6.1 Introduction

Most loops in an introductory course pair one manipulated variable with one controlled variable and one setpoint. A treadmill and the person on it do not divide that way. Belt speed is the only actuator, and the quantities that decide how fast the belt may run are blood lactate, heart rate, core temperature and the amount of carbohydrate left, none of which can be moved directly and no two of which respond on the same time scale. Raising the belt raises all of them. Which one arrives at its limit first depends on the runner rather than on the tuning, and the controller is required to hold whichever one is currently binding while the others are left wherever the actuator puts them.

That structure is the ordinary case wherever one input serves several constrained outputs. A compressor holds discharge pressure until the surge line binds, a furnace holds coil outlet temperature until a tube metal thermocouple binds, and a battery charger holds current until cell voltage or cell temperature binds. The standard answer is a selector, which computes one output per constraint and passes the most conservative of them to the actuator [1]. The part a block diagram omits is what the losing controller does while it waits, and that omission is not cosmetic. A selector without external reset winds its unselected integrators to the actuator limit, and a selector whose external reset is implemented by loading the losing integrator with the applied output deadlocks instead, because the loser then reproduces the applied output exactly and wins the next comparison by an arbitrarily small margin.

The objective adds a second layer. Distance over a fixed hour grows monotonically with the belt speed that every constraint forbids, the horizon is fixed and known, and two of the stores are worth nothing once the clock stops. An anaerobic work reserve that is still full at the finish line is distance that was available and was not taken, which makes the problem an economic optimization with a terminal condition [2, 3] rather than a regulation problem with a setpoint. A receding-horizon controller whose horizon is shorter than the run has to be told what the state it leaves behind is worth, and how that valuation is scaled turns out to decide whether it helps or hurts.

Simulators of exercise are usually built to answer physiological questions, and control studies of treadmills are usually built to hold one variable. Feedback control of heart rate on a treadmill has an established literature with identified first- and second-order models and quantified disturbance

rejection [4–6], and oxygen uptake has been controlled on the same machine [7]. However, those systems regulate one physiological variable to a reference and none of them optimizes distance, so the selector between competing limits and the value of a store at a deadline never appear. This chapter works with a one-hour treadmill in which one belt speed is driven against four physiological constraints, so that constraint selection, external reset and a terminal value can be exercised on a plant whose variables a student already has an intuition about. The physiological relations are published work with recalibrated coefficients, and the control structures are standard.

> ✦ DID YOU KNOW? **A Heart Rate Held Within Two Beats**
>
> In a 2023 laboratory study, 23 healthy participants ran 35-minute bouts on a treadmill whose speed a feedback controller set to hold a constant heart rate. The mean root-mean-square tracking error was 1.98 bpm with a second-order design and 2.13 bpm with a first-order one [8].

Section 6.2 traces closed-loop treadmills and pacing models. Section 6.3 carries every equation and every constant a reader would need to build the fourteen-state plant again, with its instrument chain. Section 6.4 measures the lags of the four constrained outputs, the static gain from belt speed to heart rate and how far the sustainable speed moves in an hour. Section 6.5 sets out the two regulators, the low selector with external reset and the receding-horizon alternative. Section 6.6 defines the scorecard, which keeps the competing costs of a run apart instead of averaging them. Section 6.7 reports the sixteen cases, from a stopped belt to a timed distance target, and measures what the terminal value is worth. Section 6.8 describes the real heart-rate straps, treadmill drives and laboratory controllers. Sections 6.9 and 6.10 state where the model stops holding and where the same structure appears in other plants.

6.2 Background: Controlling a Person on a Belt

The motor-driven treadmill entered laboratory use as an instrument for imposing a known and repeatable work rate. Graded protocols made that quantitative by stepping belt speed and grade together on a fixed schedule, and the Bruce protocol and the nomogram built from it turned the time a subject lasted into an estimate of maximal oxygen intake [9]. The arrangement is open loop by design. The schedule is fixed in advance, the subject is the disturbance, and the endpoint is the moment the subject stops. Nothing in the protocol adapts to the individual, which is what makes the test comparable across subjects and also what makes it unable to hold any one of them at a chosen physiological intensity.

Closing a loop around the belt came later and came from rehabilitation and exercise-prescription requirements, where the quantity that matters is the intensity the patient is working at rather than the speed the machine is running. Heart rate was the first controlled variable, because it is the one an inexpensive instrument reports continuously. How its models and controllers developed, from identified models to predictive control, is told in the history box of this section. Later work addressed the disturbance that dominates the loop in practice, which is very-low-frequency heart-rate variability rather than measurement noise, by shaping the input sensitivity directly [6]. Oxygen uptake has been controlled on the same machine, and it is the harder loop because the measurement is slower and noisier than heart rate [7].

What that literature does not address is the question a runner actually asks, which is how fast to go for a fixed time. Pacing has its own modeling tradition, built on an energy-flow account in which a finite anaerobic store is spent against an aerobic supply and the optimal strategy follows from how the two are scheduled over the event [3]. Those models optimize and do not regulate. The application of this chapter joins the two, putting a regulator and a receding-horizon optimizer on the same actuator and the same plant so that the difference between holding a constraint and scheduling a store can be measured on one run.

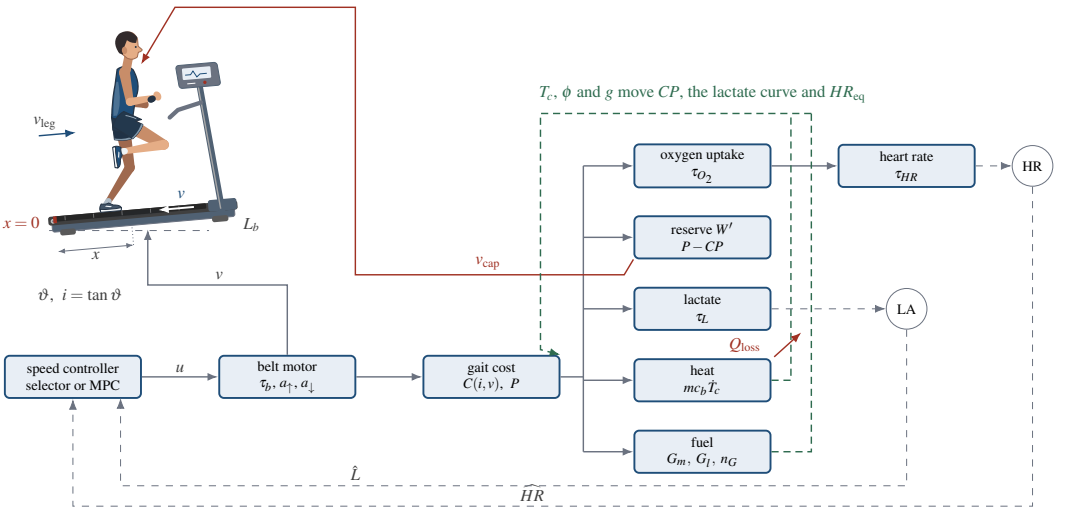


Figure 6.1 One runner on one treadmill. The belt motor follows the controller output u with lag τ_b and rate limits $a_\uparrow$ and $a_\downarrow$, the deck sits at angle ϑ of fractional grade i, and the runner stands a distance x from the back edge of a belt of usable length L_b. The legs run at v_{leg}, which equals the belt unless the belt asks for more than the capability v_{cap} the anaerobic reserve currently supports, and the run ends at $x = 0$. Gait cost $C(i, v)$ sets the metabolic demand P, which drives all five subsystems in parallel. Core temperature, accumulated fatigue and the remaining carbohydrate fraction feed back into the sustainable power, the lactate curve and the heart-rate equilibrium, drawn here as one bundle. The controller reads a noisy filtered heart rate $\widehat{HR}$ and a lactate estimate $\hat{L}$ and commands belt speed alone.

> **HISTORY** **Closing the Loop Around the Belt (2007 to 2021)**
>
> In 2007 a computer-controlled treadmill held a person's heart rate on a preset profile by changing the speed of its belt. Its designers described the heart-rate response as a static nonlinearity at the input followed by linear dynamics, identified it with a pseudorandom test signal, and compared a robust controller designed on that model with a conventional PID loop [4]. A year later a nonlinear model, identified from subjects walking on a treadmill at different speeds, carried a controller with feedforward and feedback parts that followed two different heart-rate profiles [5]. Predictive control came in 2021, on models estimated from twelve participants with a short protocol, and with the speed and acceleration of the treadmill kept within a safe range [10].

6.3 Plant Model

The plant is one runner on one treadmill for 3600 s from a standing start. Belt speed v is the only manipulated variable and is bounded at 6.7056 m/s, which is 15 mph. The deck is tilted to an angle ϑ that is either held or redrawn every 300 s from a seeded schedule. The runner occupies a position x measured from the back edge of a 1.50 m belt, and the run ends when x reaches zero. Figure 6.1 shows the arrangement and names every symbol that appears in the equations. Units are SI internally, with heart rate kept in beats per minute and lactate in mmol/L because every limit quoted for them is in those units. Display converts speed to mph, km/h and pace. The physiology below the legs, from oxygen uptake to the fuel stores, is a shared athlete model that the cycling simulator of Chapter 7 also runs. The treadmill supplies its gait cost as the metabolic demand and charges the anaerobic reserve in metabolic watts, and moving the equations into the shared model reproduced 132 recorded checkpoints of this application over 23 runs bit for bit.

The roles in Table 6.1 divide the fourteen states as follows. Two belong to the machine, being

Table 6.1 Classification of Variables for the Recreational Male Preset at Age 35 on the Level. Nominal Values Are the Threshold-Pace Steady State Where One Exists and the Measured Value After One Hour Under the Selector Otherwise.

Symbol	Role	Nominal	Range
HR	CV, measured	160 bpm	154–170 bpm band
L	CV, estimated	4.0 mmol/L	1.5–6 mmol/L SP
v	MV	2.787 m/s	0–6.7056 m/s
s	objective	9585 m	—
ϑ	DV, measured	0°	−3 to +8°
T_{air}, φ, a	DV, measured	20 °C, 0.45, 0.60	see Table 6.2
T_c	state, shown	38.57 °C	34–41.5 °C
W'	state, shown	189 m	0–190 m
$G_m + G_l$	state, shown	222 g	0–361 g
x	state, shown	0.75 m	0–1.50 m
$\dot{V}_{O_2}$	state, shown	959 W	91–1176 W
ϕ	state, hidden	0.005	0–0.40
n_G	state, shown	93.8 mmol	$\geq$ 42.2 mmol
G_u	state, shown	0 g	40 g per feed
f_R	state, shown	46.9/min	5–75/min

the belt speed and the runner's place on it, five are metabolic, four are fuel, one is thermal, one is respiratory and one is the distance integral. The fourth fuel state is carbohydrate eaten during the run and still in the gut. Only accumulated fatigue ϕ is withheld from every display, because its effect is meant to be inferred from the pace that a runner can still hold rather than read off a gauge.

6.3.1 Cost of Locomotion and Metabolic Demand

The energy cost of covering ground is taken from the measured relation for walking and running against gradient, which is a fifth-order polynomial in the fractional grade with a published level value of 3.6 J/(kg m) for running [11]. Each runner carries one economy parameter C_r that stretches the whole curve, and the running and walking polynomials are blended below 2.4 m/s so that the transient from a standing start is not an extrapolation of a running fit through a gait nobody is using, as shown in Equation (6.1).

$$\begin{aligned} C_R(i) &= 3.6 + 19.5i + 46.3i^2 - 43.3i^3 - 30.4i^4 + 155.4i^5 \\ C_W(i) &= 2.5 + 19.6i + 51.9i^2 - 76.8i^3 - 58.7i^4 + 280.5i^5 \\ w(v) &= \xi^2(3 - 2\xi), \qquad \xi = \operatorname{clamp}\left(\tfrac{v-1.6}{0.8}, 0, 1\right) \\ C(i,v) &= \max\left(0.6,\ \tfrac{C_r}{3.6}\left[0.9\,(1-w)\,C_W(i) + w\,C_R(i)\right]\right) \end{aligned} \tag{6.1}$$

where $i = \tan\vartheta$ is the fractional grade, not the angle and not a percentage, and C carries units of J/(kg m). The distinction matters numerically. At $\vartheta = 5^\circ$ the grade is 8.75 % and (6.1) returns 6.49 J/(kg m) for a runner whose level economy is 4.15 J/(kg m), which is 56 % more energy for the same meter covered.

Metabolic demand is the resting turnover plus the cost of the ground covered, as shown in Equation (6.2).

$$P = P_{\text{rest}} + m\,C(i, v_{\text{leg}})\,v_{\text{leg}}, \qquad P_{\text{rest}} = \frac{3.5\,\varepsilon_{O_2}}{60}\,m \tag{6.2}$$

where m is body mass in kg, $\varepsilon_{O_2} = 20.9\,\text{J/mL}$ is the energy equivalent of oxygen and 3.5 mL/(kg min) is one metabolic equivalent. Aerobic capacity follows the same conversion, as shown in Equa-

tion (6.3).

$$\begin{aligned} P_{\max} &= \frac{m\dot{V}_{O_2,\max}\,\varepsilon_{O_2}}{60} \\ f(P) &= \max\left(0,\ \frac{P - P_{\text{rest}}}{P_{\max} - P_{\text{rest}}}\right) \end{aligned} \tag{6.3}$$

where f is the fraction of the oxygen-uptake reserve a metabolic power represents and is the single intensity variable the rest of the model is written against. For the recreational male preset, $m = 75\,\text{kg}$ and $\dot{V}_{O_2,\max} = 45\,\text{mL}/(\text{kg}\,\text{min})$ give $P_{\text{rest}} = 91.4\,\text{W}$ and $P_{\max} = 1175.6\,\text{W}$.

6.3.2 Oxygen Uptake, Critical Power and the Anaerobic Reserve

Oxygen uptake is carried as a delivered power and follows demand with a first-order lag, saturating at the aerobic ceiling, as shown in Equation (6.4).

$$\tau_{O_2}\frac{\mathrm{d}\dot{V}_{O_2}}{\mathrm{d}t} = \min(P,\ P_{\max}) - \dot{V}_{O_2} \tag{6.4}$$

Anything demanded above $P_{\max}$ is supplied anaerobically and is accounted separately. The sustainable power is the critical-power construct, which originated as the asymptote of the work-time relation for a muscle group [12] and is now the standard two-parameter description of exercise tolerance, with a finite work reserve W' above that asymptote [13]. The application makes the sustainable power a function of the damage already done, as shown in Equation (6.5).

$$\begin{aligned} CP &= P_{\text{rest}} + f_{CP}\,(P_{\max} - P_{\text{rest}})\,(1-\phi)\,\Psi \\ \Psi &= \Psi_G\,\Psi_c, \qquad g = \frac{G_m + G_l}{G_{m,0} + G_{l,0}} \\ \Psi_G &= 0.80 + 0.20\,\text{clamp}\left(\tfrac{g}{0.15}, 0, 1\right) \\ \Psi_c &= 0.85 + 0.15\,\text{clamp}(c_G - 3,\ 0,\ 1) \end{aligned} \tag{6.5}$$

where f_{CP} is fresh critical power as a fraction of the oxygen-uptake reserve, $\phi \in [0, \phi_{\max}]$ is accumulated fatigue, g is the fraction of the starting carbohydrate store remaining and c_G is blood glucose in mmol/L. The fuel factor Ψ is unity until the store falls below 15 %, which is why an hour begun with full stores has an almost fixed sustainable power and an hour begun short of fuel does not. Peak power and the power the runner can actually produce at an instant follow in Equation (6.6).

$$\begin{aligned} P_{\text{peak}} &= P_{\text{rest}} + \mu\,(P_{\max} - P_{\text{rest}})\left(1 - \tfrac{\phi}{2}\right) \\ P_{\text{av}} &= CP + (P_{\text{peak}} - CP)\left(\frac{W'}{W'_{\max}}\right)^{0.4} \end{aligned} \tag{6.6}$$

where $\mu = 1.35$. The two-parameter model in its textbook form holds any power above CP until W' reaches zero and then drops to CP instantly [13]. That is an accurate description of time to exhaustion and an unusable one here, because a hard switch moved the runner from holding the belt to off the back of it inside two integration steps with no trend showing why. The fractional exponent in (6.6) spreads the loss of capability over most of the reserve. The reserve itself is declared as a distance D', because that is the unit it is spent in, and drains at exactly the excess power, as shown in Equation (6.7).

$$\begin{aligned} W'_{\max} &= m\,C_r\,D' \\ \frac{\mathrm{d}W'}{\mathrm{d}t} &= \begin{cases} -(P - CP), & P > CP \\ (CP - P)\,k_r\left(1 - \dfrac{W'}{W'_{\max}}\right), & P \le CP \end{cases} \end{aligned} \tag{6.7}$$

where k_r is the recovery efficiency. Above CP the time to an empty reserve is W' divided by the excess power, which is the arithmetic the belt turns into a consequence.

6.3.3 Heart Rate

Heart rate is driven by the oxygen uptake actually delivered rather than by the demand, using the equivalence of heart-rate reserve and oxygen-uptake reserve [14], with separate onset and recovery lags and two drift terms, as shown in Equations (6.8) and (6.9).

$$\begin{aligned} HR_{\mathrm{eq}} &= HR_{\mathrm{rest}} + (HR_{\mathrm{max}} - HR_{\mathrm{rest}})\, f(\dot{V}_{O_2}) \\ &\quad + k_T \max(0, T_c - 37.8) + k_F \frac{\phi}{\phi_{\mathrm{max}}} \end{aligned} \tag{6.8}$$

$$\tau_{HR} \frac{\mathrm{d}HR}{\mathrm{d}t} = \mathrm{clamp}(HR_{\mathrm{eq}},\ 35,\ 1.02\, HR_{\mathrm{max}}) - HR \tag{6.9}$$

where τ_{HR} is $\tau_{HR\uparrow}$ when HR_{eq} exceeds HR and $\tau_{HR\downarrow}$ otherwise, and $HR_{\mathrm{max}} = \mathrm{clamp}(220 - \mathrm{age} + \Delta_{HR},\ 120,\ 220)$. The two drift terms in (6.8) are the reason a constant belt speed does not produce a constant heart rate. The estimator $220 - \mathrm{age}$ is retained because every band quoted in practice is written against it, and an editable offset Δ_{HR} is provided because that estimator carries a population standard deviation near 11 bpm and systematically misestimates older adults [15]. The threshold band the application draws is 83 to 92 % of HR_{max}, which is 154 to 170 bpm at age 35.

6.3.4 Blood Lactate

Blood lactate is one well-mixed compartment whose concentration is the balance of appearance and disappearance rather than an indicator of an oxygen shortfall, which is the modern reading of lactate kinetics [16]. The steady-state concentration is a power law in the intensity above a knee, calibrated so that it balances at the resting value with the runner standing still and crosses the limit exactly at the preset's own threshold intensity, as shown in Equation (6.10).

$$\begin{aligned} L_{\mathrm{ss}}(f) &= L_{\mathrm{rest}} + (L_4 - L_{\mathrm{rest}}) \left[\max\left(0, \tfrac{f - f_0}{f_{T,\mathrm{eff}} - f_0}\right) \right]^q \Lambda \\ \tau_L \frac{\mathrm{d}L}{\mathrm{d}t} &= L_{\mathrm{ss}}(f(P)) - L \end{aligned} \tag{6.10}$$

where f_0 is the intensity below which lactate does not rise, q places the knee, $\Lambda = 0.6 + 0.4\,\mathrm{clamp}(g/0.2, 0, 1)$ carries the effect of a depleted store and $f_{T,\mathrm{eff}} = \max\big(0.40,\ f_T(1-\phi)\Psi\big)$ is the intensity at which the curve crosses the limit, with f_T the fresh lactate marker. The marker is scaled by the same fatigue and fuel factors as critical power in (6.5), so that a tired runner's lactate does not read like a fresh one's. The constraint $L_4 = 4\,\mathrm{mmol/L}$ is the conventional working threshold [17]. The marker f_T and the critical-power fraction f_{CP} are separate parameters, because critical power and the lactate threshold markers correlate closely without being interchangeable [18], and every preset sets them equal. For the presets the lactate constraint and the reserve constraint are then one constraint, which is the teaching simplification that keeps the selector legible, and editing either parameter separates them.

> ⧖ HISTORY **Four Millimoles on the Treadmill (1976 to 1985)**
>
> The lactate limit of 4 mmol/L has a laboratory history. In 1976 a group of physiologists in Cologne placed the change from purely aerobic to partially anaerobic metabolism at an earlobe blood lactate of 4 mmol/L, without data to support it. From 1979 the group tested the claim on a treadmill. Each participant ran 25 min at one constant speed, stopping for 30 s every 5 min for an earlobe sample, and several such runs near the threshold were needed. In 15 participants lactate stayed constant at intensities where it reached 4.0 ± 0.7 mmol/L. The data appeared in German in 1982, and the English term maximal lactate steady state followed in 1985. The test is too long for routine testing and remains the gold standard against which quicker tests are checked [19].

6.3.5 Fatigue, Heat and Respiration

Accumulated fatigue is the mechanism by which sustained lactate and sustained thermal strain reduce the sustainable power in (6.5), as shown in Equation (6.11).

$$\begin{aligned} \Gamma &= k_L \frac{\max(0, L - L_4)}{L_4} + k_{TF} \max(0, T_c - 38.5) \\ \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \Gamma \left(1 - \frac{\phi}{\phi_{\max}}\right) - k_\phi \, \phi \, \mathrm{clamp}\left(1 - \tfrac{P}{CP}, 0, 1\right) \end{aligned} \tag{6.11}$$

The thermal term is present because a rising core temperature reduces the work a person can sustain independently of the metabolic state [20]. The heat balance itself is one lumped body node written about the resting equilibrium, as shown in Equation (6.12).

$$\begin{aligned} m\, c_b \frac{\mathrm{d}T_c}{\mathrm{d}t} &= \max(0,\ P - P_{\mathrm{mech}}) - Q_{\mathrm{loss}} \\ Q_{\mathrm{loss}} &= P_{\mathrm{rest}} + G_d \,(T_c - 37.0) \\ &\quad + \min\left(G_e (T_c - 37.05),\ E_{\max}\right) \end{aligned} \tag{6.12}$$

where $P_{\mathrm{mech}} = m\, g\, v_{\mathrm{leg}} \sin\vartheta$ for $\vartheta > 0$ is the mechanical work leaving the balance against gravity, and c_b is the specific heat of the body. Writing the loss term about 37.0 °C makes a standing runner in a 20 °C room exactly steady rather than drifting, which is verified as an equality in the application's own test suite. The conductances scale with the runner and the room, as shown in Equation (6.13).

$$\begin{aligned} \Phi &= \frac{m \dot{V}_{O_2,\max}}{75 \times 45} \\ G_d &= (g_0 + g_a a) \frac{m}{75} \, \mathrm{clamp}\left(1 + \tfrac{20 - T_{\mathrm{air}}}{25}, 0.3, 1.8\right) \\ G_e &= k_s \, \Phi \, (1 - 0.55\varphi)(0.45 + 0.55a) \\ E_{\max} &= E_0 \, \Phi \, (1 - 0.6\varphi)(0.45 + 0.55a) \end{aligned} \tag{6.13}$$

where $a \in [0, 1]$ is airflow over the runner, φ is relative humidity and Φ is aerobic capacity relative to the default preset. Both sweating terms scale with Φ because an elite runner produces about 1.6 times the heat of a recreational runner in a smaller body, and a fixed evaporative ceiling left that runner with no thermal steady state at their own threshold pace. Respiratory frequency is a first-order response to intensity with additive terms in lactate and temperature, as shown in Equation (6.14), and is displayed rather than controlled.

$$\begin{aligned} f_{R,\mathrm{eq}} &= f_{R,0} + k_R f(\dot{V}_{O_2}) \\ &\quad + 2\max(0, L - 2) + 3\max(0, T_c - 38) \\ \tau_R \frac{\mathrm{d}f_R}{\mathrm{d}t} &= \min(70,\ f_{R,\mathrm{eq}}) - f_R \end{aligned} \tag{6.14}$$

6.3.6 Substrate Selection and the Carbohydrate Stores

Fat and carbohydrate are oxidized concurrently with a split that moves with intensity, and fat oxidation carries a rate ceiling that carbohydrate does not [21]. The split and the three fuel states are given by Equations (6.15) and (6.16).

$$\begin{aligned} \chi &= \mathrm{clamp}(0.20 + 0.78 f,\ 0.15,\ 0.98) \\ P_F &= \min\left(P_{F,\max}\left[0.7 + 0.3\left(1 - \tfrac{G_m}{G_{m,0}}\right)\right],\ (1 - \chi) P_{\mathrm{del}}\right) \\ P_C &= P_{\mathrm{del}} - P_F, \qquad \dot{n}_C = \frac{P_C + P_{\mathrm{an}}}{e_G} \end{aligned} \tag{6.15}$$

where $P_{\text{del}} = \min(P, P_{\max})$, $P_{\text{an}} = \max(0, P - P_{\text{del}})$ is the anaerobic contribution, and $e_G = 3153\,\text{J/mmol}$ is the metabolizable energy of a millimole of glucose equivalent. Counting P_{an} as carbohydrate is what keeps the anaerobic reserve from being free energy.

$$
\begin{aligned}
\beta &= \operatorname{clamp}\Big(0.18 + 0.35\left(1 - \tfrac{G_m}{G_{m,0}}\right), 0,\ 0.6\Big) \\
\frac{\mathrm{d}G_m}{\mathrm{d}t} &= -\dot{n}_C(1-\beta)M_G \\
\frac{\mathrm{d}G_l}{\mathrm{d}t} &= -\dot{n}_H M_G, \qquad \frac{\mathrm{d}G_u}{\mathrm{d}t} = -\frac{G_u}{\tau_u} \\
\frac{\mathrm{d}n_G}{\mathrm{d}t} &= \dot{n}_H + \dot{n}_g - \dot{n}_C\beta + \frac{G_u}{\tau_u M_G} \\
&\quad - k_u \max\left(0,\ \frac{n_G}{V_G m} - c_{G,0}\right)
\end{aligned}
\tag{6.16}
$$

where $M_G = 0.180\,16\,\text{g/mmol}$, β is the share of the carbohydrate flux drawn from circulating glucose and rises as muscle glycogen falls, and the endogenous supply is gluconeogenesis first and hepatic glycogenolysis for the remainder, as shown in Equation (6.17).

$$
\begin{aligned}
\dot{n}_w &= \max\big(0,\ \dot{n}_C\beta + 0.02(c_{G,0}V_G m - n_G)\big) \\
\dot{n}_g &= \min(k_g,\ \dot{n}_w) \\
\ell &= \operatorname{clamp}\Big(\tfrac{G_l}{0.15\,G_{l,0}},\ 0,\ 1\Big) \\
\dot{n}_H &= \operatorname{clamp}(\dot{n}_w - \dot{n}_g,\ 0,\ \dot{n}_{H,\max}\,\ell)
\end{aligned}
\tag{6.17}
$$

Writing gluconeogenesis as an unconditional source instead let blood glucose climb while the runner stood still, which is an unclosed balance rather than a teaching point. Carbohydrate eaten during the run enters the gut store G_u and empties into the circulating pool with $\tau_u = 600\,\text{s}$, and circulating glucose above its starting concentration is taken up at $k_u = 0.12\,\text{L/s}$ per unit of excess. This sink stands in for the hormonal response without modeling it. Its only job is to stop a meal from piling up in the small circulating pool, and it does nothing while glucose sits at or under the starting concentration, which is the case for the whole of every scenario without a feed. Glucose it clears is burned instead of fat, and since fat is not a tracked store the carbohydrate accounting still closes. Each feed places 40 g in the gut. In prolonged exercise a carbohydrate feed keeps plasma glucose from falling without sparing muscle glycogen [22], and the gut store behaves the same way. The store is calibrated so that level running at threshold pace exhausts it after a documented distance, which is what the fuel gauge reads at full scale, and the exhaustion distance is checked against the integrated model rather than asserted. Progressive glycogen depletion at a fixed work rate and the pace decline that follows it are the behavior this reproduces [22].

6.3.7 Belt, Position and the Two Moving Ceilings

The belt is a first-order motor with separate acceleration and deceleration limits, and the runner's position on it is the state that ends the run, as shown in Equation (6.18).

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= \operatorname{clamp}\left(\frac{u - v}{\tau_b},\ -a_{\downarrow},\ a_{\uparrow}\right), \qquad v \in [0,\ v_{\max}] \\
v_w &= v + \operatorname{clamp}\big(k_c(x_0 - x),\ -0.25,\ 0.7\big) \\
v_{\text{leg}} &= \max\big(0,\ \min(v_w,\ v_{\text{cap}})\big) \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \operatorname{clamp}(v_{\text{leg}} - v,\ -3,\ 3), \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = v_{\text{leg}}
\end{aligned}
\tag{6.18}
$$

where v_{cap} is the speed corresponding to P_{av} of (6.6) on the grade underfoot, k_c is how hard the runner works to stay centered, and x_0 is the starting position. When the belt asks for more than the runner can produce, v_{leg} falls short of v, the runner travels backward along the deck, and the run ends at $x = 0$. The failure mode is therefore a position and not an overshoot on a trend.

> **TRY IT IN THE APP** **2 · A Speed Step by Hand**
>
> Load scenario 2 and press Run, then watch the runner's position on the belt. The belt steps to 9.36 km/h at 60 s and everything settles. At 15 min it steps to 11.16 km/h, above critical power, and nothing settles: the runner drifts back along the deck and goes off the back of the belt at 24:28 with 3.93 km covered.

Both scored ceilings open over a finishing window, as shown in Equation (6.19), because a reserve that is still full when the clock stops was distance that was available and not taken.

$$\begin{aligned} \zeta &= \mathrm{clamp}\left(\frac{t-(T-t_w)}{t_w},\ 0,\ 1\right) \\ L_{\mathrm{lim}}(t) &= L_4 + (L_f - L_4)\,\zeta^2 \\ HR_{\mathrm{lim}}(t) &= HR_{\mathrm{hi}} + (HR_{\mathrm{max}} - HR_{\mathrm{hi}})\,\zeta^2 \end{aligned} \tag{6.19}$$

With $T = 3600\,\mathrm{s}$ and $t_w = 300\,\mathrm{s}$, the lactate ceiling holds at 4.0 mmol/L for 55 min and reaches 6.0 mmol/L at the finish, and the heart-rate ceiling rises from 170 bpm to 185 bpm over the same window at age 35. The quadratic shape gives the ceiling a continuous value and slope where it starts to open.

6.3.8 Parameters, Presets and Numerics

Table 6.2 lists the constants of the default preset. Every one of them is adjustable in the application and all but one are displayed. Four presets write mass, aerobic capacity, economy, the lactate marker, the critical-power fraction, the reserve, the lag constants and the fuel stores in one action, and the level threshold pace each of them implies is measured from the model rather than asserted at it, at 10.03 km/h for the recreational male preset, 8.72 km/h for the recreational female preset, 20.26 km/h for the elite male preset and 17.20 km/h for the elite female preset. The relative intensity at 4 mmol/L is deliberately not given a difference between the male and female presets.

The engine of Section 1.6.1 advances the state vector by fourth-order Runge-Kutta at a fixed 0.25 s step. The regulators execute once per simulated second, the optimizer replans once per five, and the historian keeps one sample per second. Two independent seeded streams supply the randomness, one for the incline schedule and one for the instrument noise, so turning the noise up leaves the hills exactly where they were. The playback multipliers are 1, 5, 20 and 60.

6.3.9 Assumptions

The model is a teaching representation of a control problem. It is not a training plan, not a fitness assessment and not medical advice, and no number in it describes any individual. Each simplification fails somewhere.

Critical power and the 4 mmol/L intensity are equal in every preset. They are close in practice and are not identical, and neither construct has a single agreed determination [13, 17]. Setting them equal collapses the lactate constraint and the reserve constraint into one constraint for the presets, which is the simplification that makes the selector legible. The two are separate parameters, so a student who edits one can study the difference, and the presets cannot show it on their own.

Lactate is one well-mixed compartment. There is no muscle-to-blood transport delay, no shuttling between fiber types and no distinction between arterial and venous sampling, all of which are real and all of which would make the measurement later than the 250 s already modeled [16]. The application understates how far behind the legs the measurement is.

Table 6.2 Model Parameters of the Recreational Male Preset. All Are Adjustable. Only Accumulated Fatigue Is Withheld From Every Display.

Symbol	Meaning	Value	Unit
m	body mass	75	kg
age	age	35	years
$\dot{V}_{O_2,\max}$	maximal oxygen uptake	45	mL/(kg min)
C_r	level running economy	4.15	J/(kg m)
f_T	intensity at 4 mmol/L	0.80	—
f_{CP}	critical power, fraction of reserve	0.80	—
D'	anaerobic distance reserve	190	m
HR_{rest}	resting heart rate	60	bpm
Δ_{HR}	offset from 220 − age	0	bpm
τ_{O_2}	oxygen-uptake lag	35	s
$\tau_{HR\uparrow}$	heart-rate onset lag	45	s
$\tau_{HR\downarrow}$	heart-rate recovery lag	70	s
τ_L	lactate clearance	250	s
τ_R	respiratory lag	20	s
k_T	drift with core temperature	7	bpm/°C
k_F	drift at full fatigue	10	bpm
f_0	lactate knee	0.35	—
q	lactate curvature	3.0	—
k_r	reserve recovery efficiency	0.5	—
μ	peak power multiple	1.35	—
k_L	fatigue from lactate	1.7×10^{-4}	1/s
k_{TF}	fatigue from heat	4.6×10^{-5}	1/(s °C)
k_ϕ	fatigue recovery	2.0×10^{-4}	1/s
$\phi_{\max}$	largest power loss	0.40	—
T_{air}	room temperature	20	°C
φ	relative humidity	0.45	—
a	airflow	0.60	—
c_b	specific heat of the body	3500	J/(kg K)
k_s	evaporative gain	900	W/K
E_0	evaporative ceiling	1820	W
g_0	dry loss in still air	16	W/K
g_a	extra dry loss at full airflow	22	W/K
$G_{m,0}$	working-muscle glycogen	295	g
$G_{l,0}$	liver glycogen	66	g
$c_{G,0}$	starting blood glucose	5.0	mmol/L
V_G	glucose distribution volume	0.25	L/kg
k_g	gluconeogenesis	0.02	mmol/s
$\dot{n}_{H,\max}$	largest hepatic output	0.18	mmol/s
$P_{F,\max}$	fat oxidation ceiling	420	W
τ_u, k_u	gastric emptying, glucose uptake	600, 0.12	s, L/s
$v_{\max}$	top belt speed	6.7056	m/s
$a_\uparrow$, $a_\downarrow$	belt rate limits	0.35, 0.50	m/s^2
τ_b	belt motor lag	0.6	s
L_b, x_0	belt length, start position	1.50, 0.75	m
k_c	recentering effort	0.5	1/s
L_4, L_f	lactate constraint, finish	4.0, 6.0	mmol/L
t_w	finishing window	300	s
T	length of the run	3600	s

Heart rate responds to intensity, temperature and fatigue only. Hydration, posture, caffeine, altitude and the last meal do not exist here, and the very-low-frequency variability that dominates a real heart-rate control loop is represented only as filtered measurement noise [6]. The estimator 220 − age places an individual's band up to ten beats either side of where the application draws it [15].

Running economy is one coefficient. Real economy varies with speed, surface, footwear and fatigue

Table 6.3 Response to a Belt Step From Rest to 9.4 km/h at $t = 60$ s, Recreational Male Preset on the Level. Final Values Are Read at 60 min, and Times From the Step to 63 % of the Total Change Are Read at the 0.25 s Integration Step.

Output	Initial	Final	t_{63} (s)
Oxygen uptake (mL/(kg min))	3.50	34.48	40
Heart rate (bpm)	60.0	158.2	96
Blood lactate (mmol/L)	1.00	3.05	257
Core temperature (°C)	37.00	38.50	475

in ways a single multiplier on the gradient curve cannot carry, and (6.1) is a fit to treadmill data at steady speeds [11].

The thermal model has one body node. There is no skin compartment, no sweat-rate dynamics, no dehydration and no cardiovascular consequence of fluid loss, so a hot hour goes wrong more slowly here than it does in practice [20]. The coefficients in (6.13) are teaching calibrations and are not validated fits.

Feeding is carbohydrate alone. A feeding event empties from the gut with one time constant, glucose above the resting value is removed by a proportional sink rather than an insulin response, and there is no drink and no refilling of muscle glycogen during the run, so a fed hour postpones the fall in sustainable pace rather than removing it [22].

Falling off the back is modeled as a position reaching zero. No real treadmill permits that, because a safety key stops the belt first. The state is in the model because the consequence of demanding more than the runner can produce has to land somewhere other than a performance metric.

Repeatability is unrealistic. Every run of a given scenario returns the same numbers to the last digit, which is convenient for marking and is something no physiological experiment has ever managed.

6.4 Dynamic Analysis

6.4.1 Four Lags in Series on One Actuator

The distinctive dynamic feature of this plant is that one input reaches four constrained outputs through lags that differ by an order of magnitude. Table 6.3 reports the time each output takes to complete 63 % of its total change after the belt is stepped from rest to 9.4 km/h, measured in the application from a single run, and Figure 6.2 sets them on one time axis.

The ordering in Table 6.3 is the control problem. The legs are doing the new work within a second, the fastest instrument that reports anything is 96 s behind, the constraint that actually binds is 257 s behind, and the constraint with no actuator behind it is 475 s behind. By the time the fast variable reports that the belt went too high, the slow one has already begun to charge for it. A loop closed on lactate cannot be faster than 257 s without chasing a measurement that has not happened yet, and a loop closed on heart rate sees 35 s and 45 s of physiological lag in series with a 6 s measurement filter.

6.4.2 Static Gain From Belt Speed to Heart Rate

Linearizing (6.2), (6.3) and (6.8) at a steady speed above the walking blend gives the static gain in Equation (6.20), because C is independent of speed once $w = 1$.

$$\frac{\partial HR}{\partial v} = \frac{HR_{\max} - HR_{\text{rest}}}{P_{\max} - P_{\text{rest}}}\, m C_r \tag{6.20}$$

Equation (6.20) evaluated at the recreational male preset's threshold speed of 2.787 m/s gives $125/1084.2 \times 75 \times 4.15 = 35.9$ bpm per m/s, confirmed by a central difference on the model at 35.9 bpm per m/s. The same expression gives 19.3 bpm per m/s for the elite male preset and 41.3 bpm per m/s for the recreational female preset, a spread of 2.1 across the four presets. Mass and economy

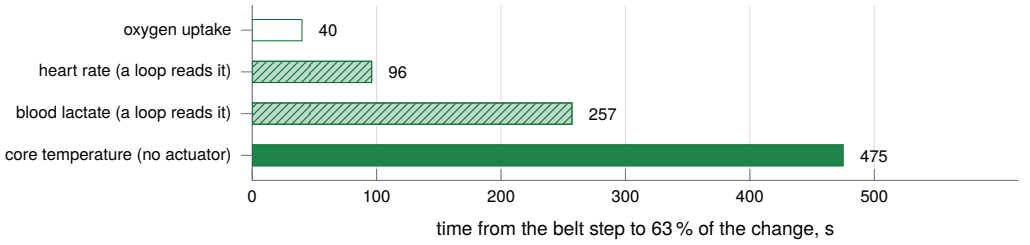


Figure 6.2 The four lags of Table 6.3 on one time axis, after a belt step from rest to 9.4 km/h. Hatched bars are the two outputs the loops read, the white bar is oxygen uptake and the solid bar is core temperature. Notice that the constraint that binds, lactate, answers 257 s after the belt moves, and a loop closed on it cannot be faster than that without chasing a measurement that has not happened yet.

Table 6.4 Movement of the Sustainable Speed Over One Hour, Recreational Male Preset Under the Selector, Read at 10 min and 60 min.

Condition	v_{sus} (km/h)	Change	T_c (°C)	$\phi / \phi_{\max}$
Level, 20 °C, fueled	10.03 → 9.98	−0.5 %	38.57	1 %
28 °C, 60 % RH, $a = 0.15$	10.03 → 8.88	−11.4 %	40.26	29 %
One third of the stores	10.03 → 8.00	−20.2 %	38.30	0 %

enter the numerator and aerobic capacity enters the denominator, so a fitter runner in a lighter body presents a plant gain less than half that of a recreational runner. A fixed controller gain in m/s per bpm therefore cannot serve both, which is the reason the tuning is scheduled.

6.4.3 When the Constraint Moves and When It Does Not

The sustainable speed is the speed at which (6.10) settles exactly on the limit, and it moves only through ϕ and Ψ in (6.5). Table 6.4 measures how far it actually travels over an hour in three conditions, read at 10 min and at 60 min.

Table 6.4 corrects an intuition that the application is often used to test. On a level hour in a cool room with full stores the constraint is very nearly fixed, falling 0.5 % over the whole hour, because the fatigue term reaches only 1 % of its ceiling and the fuel factor Ψ in (6.5) stays at unity while the store remains above 15 %. The constraint moves when a store is actually being exhausted. In the warm room the thermal term in (6.11) takes 29 % of the sustainable power and the sustainable speed falls 11.4 %, and starting with one third of the stores costs 20.2 % with the core temperature almost unchanged. The two mechanisms are separable on the trends and they are not separable on the pace, which is the diagnostic point.

The under-fueled hour also shows that the mechanism is a rate ceiling and not a shortage of energy. The belt averages 9.69 km/h from 10 to 20 min, exactly as it does with full stores, with the gauge falling steadily and nothing else out of range. At 30 min the muscle store has reached 57 g, blood glucose has left 5.00 mmol/L and the sustainable speed has begun to fall. By 35 min the belt is at 8.50 km/h and blood glucose is at 2.55 mmol/L. Fat oxidation continues throughout and the fat store is effectively unbounded, but the ceiling $P_{F,\max}$ in (6.15) caps what it can deliver per second, so the pace falls with the store and does not recover inside the hour.

6.5 Controller Design

Three modes drive the same actuator. Manual passes an operator command through to the belt. The regulatory mode runs two proportional-integral loops through a low selector. The predictive mode replaces both with a receding-horizon search. All three share the actuator handling and the instrument chain.

6.5.1 Instruments

The heart-rate measurement carries colored noise and a first-order filter of 6 s, as shown in Equation (6.21).

$$\begin{aligned} \widehat{HR}_k &= \alpha_H \widehat{HR}_{k-1} + (1-\alpha_H)\,(HR + \sigma_H n_k) \\ \alpha_H &= \mathrm{e}^{-\Delta t_c/\tau_H} \end{aligned} \tag{6.21}$$

where σ_H defaults to 0.8 bpm and n_k is drawn as white noise and then passed through a low-pass filter of its own, with the amplitude rescaled afterward so that the stationary standard deviation of the result is still unity. Advancing α_H by the exponential rather than by a forward difference keeps the filter stable at any sample period. The lactate signal is produced the same way with a 20 s filter and a default error of 0.06 mmol/L, and it is labeled on every display as an estimate. There is no continuous blood-lactate sensor and no glycogen sensor, and the application does not imply either instrument.

6.5.2 Two Regulators and a Low Selector

Both loops produce a belt speed and both are bounded by the belt. Each is the shared incremental, or velocity, form algorithm, with its derivative acting on the measurement rather than the error, as shown in Equation (6.22).

$$\begin{aligned} e_k &= \mathrm{SP}_k - \mathrm{PV}_k, \qquad p_k = K_c e_k \\ d_k &= -K_c \tau_D \frac{\mathrm{PV}_k^f - \mathrm{PV}_{k-1}^f}{\Delta t_c} \\ s_k &= s_{k-1} + (p_k - p_{k-1}) + (d_k - d_{k-1}) \\ c_k &= u_b + s_k + I_{k-1} \\ u_k &= \mathrm{clamp}(c_k,\ 0,\ v_{\max}) \\ I_k &= I_{k-1} + \left[\frac{K_c}{\tau_I} e_k + \frac{u_k - c_k}{T_t}\right] \Delta t_c \end{aligned} \tag{6.22}$$

where u_b is the bias and PV^f is the filtered measurement. These are Equations (1.12) to (1.15) of Section 1.6.2 in the loop's own symbols, with $s_k = S_k$ and $c_k = v_k$, and anti-windup is back-calculation on I_k alone with the tracking constant $T_t = \tau_I$. Both loops are reverse acting, $a = +1$ in the convention of Section 1.3, because a faster belt raises heart rate and lactate. The increments difference the fully formed terms, so a gain changed in automatic moves the output at once by $(K_{c,\mathrm{new}} - K_{c,\mathrm{old}})\, e_k$. That change is deliberately not bumpless. Restoring the scenario 6 heart-rate loop from 11 to 1.1 % and from 30 to 150 s at 15 min, with the loop 59 bpm below its reference, stops the belt within 5 s and holds it stopped for 243 s. In manual, I_k is solved backward from whatever the operator has set, which makes both directions of transfer bumpless. The heart-rate loop uses $K_c = 1.1\,\%$ of threshold pace per bpm, $\tau_I = 150\,\mathrm{s}$ and $\tau_D = 0$. The lactate loop uses $K_c = 20\,\%$ of threshold pace per mmol/L and $\tau_I = 420\,\mathrm{s}$. Derivative action defaults to zero on both, because Table 6.3 shows nothing fast for a derivative to anticipate and (6.21) supplies noise for it to amplify.

Both gains are carried as a fraction of the runner's own nominal threshold pace and multiplied by it at execution, which is gain scheduling on the one number that says how fast this runner operates. Equation (6.20) gives a factor of 2.1 in plant gain across the four presets against a factor of 2.3 in operating speed, and the schedule is what allows one tuning to serve all of them.

Both loops also trim around a feedforward rather than building their entire output from an integral, as shown in Equation (6.23).

$$u_b = v_{\mathrm{nom}}\, \mathrm{clamp}\left(\frac{t}{t_r/2}, 0, 1\right) \frac{C(0, v_{\mathrm{nom}})}{C(i, v_{\mathrm{nom}})} \tag{6.23}$$

where v_{nom} is the runner's nominal level threshold pace computed from the selected preset and t_r is the reference ramp length. The second factor is the optional incline feedforward, which converts a

level-ground speed into the speed that costs the same energy per second on the grade underfoot. It is a measured feedforward and not a predicted one, because the angle is available the instant it changes while the feedback path behind it is 257 s long.

The selector and its external reset are given in Equation (6.24).

$$\begin{aligned} u &= \min(u_{HR},\ u_L) \\ &\text{track } u_j \text{ to } u+\delta \text{ whenever } u_j > u+\delta \end{aligned} \tag{6.24}$$

where j indexes the loop that was not selected and $\delta = 0.05\,\mathrm{m/s}$, which is 0.7 % of the belt's range. The band in (6.24) is the part that is easy to get wrong. Tracking the losing loop to exactly u puts its next output exactly on u, where it wins the following comparison by an arbitrarily small margin and discards the winner's integral increment before it reaches the belt. The two loops then exchange the selection every scan and the actuator does not move. During development that fault held the belt at a walking pace for a whole hour with an 86 bpm heart-rate error and both trends looking reasonable. Letting the losing loop sit up to δ above the applied output leaves it free to fall through the selection, which is the entire purpose of a selector, while preventing it from winding to the actuator limit. Volume 4, Chapter 4 shares a ground speed among several demands through the same structure, with an incumbent margin that keeps two nearly equal demands from trading the drive.

🕮 **Pause.** Try Exercise 6.1 before reading on.

A lost heart-rate signal changes the selection rather than the loops. The monitor holds its last reading, the heart-rate request leaves the comparison, and a cap at 85 % of the nominal threshold pace takes its place, so that $u = \min(u_L,\ 0.85\,v_{\mathrm{nom}})$ until the signal returns. A frozen reading is not a measurement, and a loop that kept acting on it would hold the runner to a heart rate that is no longer there. The cap is a speed the runner can hold by construction.

6.5.3 Receding-Horizon Alternative

The predictive controller is the application's own nonlinear search and not the linear option of Section 1.6.3. It replans every 5 s over a horizon of 300 s, integrating a reduced copy of the selected preset forward on a 5 s step from its own estimate of the runner. The reduction drops respiration, which constrains nothing, carries blood glucose without the gut store or the uptake sink, and replaces the 0.6 s motor lag with a rate limit, because 0.6 s of motor against a 5 s prediction step is not worth integrating. The decision variables are four blocked speeds with block edges at 0, 10, 30 and 60 % of the horizon. The objective is Equation (6.25).

$$\begin{aligned} J = s_N + \Phi(\mathbf{z}_N) + \tfrac{1}{2}\frac{W'_N}{mC_r}\,\mathrm{clamp}\!\left(\frac{T-t_N}{300\,\mathrm{s}},0,1\right) \\ - 20\lambda\sum_k |u_k - u_{k-1}| - 12\,\Omega \end{aligned} \tag{6.25}$$

where s_N is the distance predicted inside the horizon, $\mathbf{z}_N$ is the predicted end state, λ is a move penalty and Ω accumulates weighted constraint violation. The last two of the leading terms are the terminal value. The first of them, Φ, is the distance still available after the horizon, obtained by rolling the reduced plant forward from $\mathbf{z}_N$ under the base policy of holding sustainable pace on the grade underfoot, in at most sixteen steps of nominally 150 s, as shown in Equation (6.26). This Φ is not the capacity ratio of Equation (6.13).

$$\Phi(\mathbf{z}_N) = \sum_{j=1}^{n} v_{\mathrm{sus}}\big(\mathbf{z}_{N+j-1}\big)\,\Delta t_j, \qquad \sum_j \Delta t_j = T - t_N \tag{6.26}$$

The stores drain along that rollout as they would in the plant, which is what distinguishes Φ from the product $(T-t_N)\,v_{\mathrm{sus}}(\mathbf{z}_N)$ that the same idea suggests. Section 6.7.8 measures the difference and it

Table 6.5 The Application's Own Performance Measures. All Are Accumulated at the 0.25 s Integration Step Over the Whole Run and Are Frozen When the Run Ends.

Measure	Definition	What it is for
Distance	$s(T)$	the objective
Average pace	s/t including the start	the unit a runner reads
Compliant	both integrated excesses below 1 and 10	one pass or fail over both ceilings
Lactate excess	$\int_0^T \max(0, L - L_{\text{lim}})\,\mathrm{d}t$	area above a moving ceiling
Above the ceiling	time with $HR > HR_{\text{lim}}$	duration rather than area
Peak core	$\max_t T_c$	the constraint with no actuator
Fuel left	$F\,g(T)$ in equivalent miles	what the hour cost in store
Reserve left	$W'(T)/(mC_r)$ in meters	whether the deadline was used
Min margin	$\min_t x$	how close the run came to ending

is not small. The second terminal term credits the reserve the base policy never spends, at its own length in meters, ramped to zero over the last 300 s so that a reserve still held at the finish line is worth nothing, which is where the finishing effort comes from. That ramp is a fixed 300 s, equal to the default t_w, and does not follow an edited finishing window.

Constraint violation is evaluated against margins inside the true limits, at 0.25 mmol/L inside the lactate ceiling and 2 bpm inside the heart-rate ceiling, to cover the hill the controller cannot see coming. Candidates are compared on feasibility first and on J second, so a plan that violates less always beats a plan that goes further and the application never softens a constraint to buy distance. The search is a bounded coordinate refinement over deltas of 0.60, 0.22 and 0.08 m/s from four warm starts, capped at 260 plan evaluations. Over the flat hour it uses a mean of 37 evaluations per replan with a maximum of 68, and the mean solve time including the rollout is 0.56 ms in a local Chromium measurement with a maximum of 3.5 ms against a replan period of 5 s. When no feasible plan exists, the search still returns the least-violating one and the applied speed is also limited to the nominal sustainable pace, which is survivable by construction, and the application reports the event rather than hiding it.

In both automatic modes the command itself is rate limited to 1.5 times the belt's own acceleration and deceleration limits, which makes every transfer between modes and architectures bumpless by construction, including the transfer from predictive to regulatory control that no optimizer warm start guarantees on its own.

6.6 Performance Measures

The application reports nine measures beside the per-loop integral absolute error, overshoot, settling time, actuator travel and time saturated that the shared framework computes for every declared loop at the logging rate (Section 1.6.4). The set is deliberately a list of competing costs rather than one composite score, and Table 6.5 gives each one with what it is computed from and what it is for.

Four properties of that set are worth stating because they are design decisions rather than conventions. Compliance is judged on integrated excess rather than on any excursion at all, so that a sample or two of numerical overshoot at a constraint the run is deliberately riding does not read as a violation and a genuine excursion cannot hide behind a short duration. The lactate measure is an area and the heart-rate measure is a duration, because the lactate ceiling is a hard constraint whose magnitude matters and the top of the heart-rate band is a target whose duration matters. Reserve left is reported without a pass band until the run ends, because a full reserve is correct at 20 min and is a failure at 60 min. The distance and the costs are shown together and are never combined, so that a faster run and a compliant run can be compared on the record rather than through a weighting nobody agreed to.

The integral absolute error the framework reports for the heart-rate loop accrues for the whole 3600 s and includes the approach from rest, so it is quoted here only as a comparison between runs of identical length and never as a bare number. Actuator travel, here called command travel, is the total variation $\sum_k |u_k - u_{k-1}|$ of the speed command in m/s, which the application records with the run. The framework's own travel figure differences the logged belt speed instead and reads slightly lower. Command travel is the measure that separates a controller buying distance with continuous speed changes from one that is not.

6.7 Results

The sixteen cases the application ships with are listed in Table 6.6, ordered the way a course would take them, with the mechanism each exercises and what completes it. Loading one writes the mode, the architecture, the runner preset, the tuning, the environment and the event schedule together, and states in advance what the run is meant to show. Completion and the certificates work as Section 1.6.4 describes. Reset restarts the run from $t = 0$ and keeps every setting the learner changed, which is how the before-and-after comparisons are made, while reloading the case restores its own settings. The challenge is judged on the case as given, so changing a model parameter or a disturbance voids the attempt.

6.7.1 Manual Operation and the Failure Mode

Scenario 2 makes the plant's character clear before any controller is involved. The belt steps to 9.36 km/h at 60 s, which is 0.67 km/h below the recreational male preset's threshold pace of 10.03 km/h, and everything settles with lactate at 2.98 mmol/L at 15 min. The belt then steps to 11.16 km/h, which is above critical power, and nothing settles at all. Lactate reaches 4.62 mmol/L three minutes later and 6.10 mmol/L at 24 min, the reserve drains at the excess power of (6.7), and the runner reaches the back edge of the belt at 24:28 with 3.93 km covered. The same experiment at four constant speeds gives failures at 52 s at 16.2 km/h, 261 s at 12.6 km/h and 577 s at 11.2 km/h, the last with 1.78 km covered, and a completed hour of 9.39 km at 9.4 km/h. A speed 11 % above threshold is not an offset that a controller trims out. It is a countdown whose length is the reserve divided by the excess.

6.7.2 Gradient Disturbances and Measured Feedforward

Scenario 4 steps the deck to 5° at 10 min with the controller settled. The belt speed does not change at that instant and the energy every meter costs rises 56 %, so the disturbance is multiplicative on the plant gain and on the operating point at once. Table 6.7 gives the measured result with and without the incline feedforward of (6.23).

> **TRY IT IN THE APP 4 · A Hill Arrives**
>
> Load scenario 4 and press Run with Incline feedforward off. At 10 min the deck tilts to 5°, and lactate peaks at 4.52 mmol/L, past its 4.0 mmol/L limit. Press Reset, open Advanced, set Incline feedforward to On and press Run again. The peak is now 3.50 mmol/L and the lactate excess is zero.

The feedforward acts at the instant the angle changes and the feedback path behind it is 257 s long, so the single hill of scenario 4 costs 29.2 mmol s/L of integrated excess with feedback alone and nothing at all with the feedforward on, while the peak falls from 4.52 to 3.50 mmol/L. Repeating that disturbance twelve times over an hour in scenario 10 multiplies the difference, at 200.3 mmol s/L of excess and a peak of 5.32 mmol/L with feedback alone against zero excess and a peak of 3.86 mmol/L with the feedforward. The shipped noise seed sits at the top of the range for feedback alone. Five other instrument-noise seeds on the same course give 11.7 to 27.5 mmol s/L for the single hill and 175 to 196 mmol s/L for the hour, and the feedforward holds the excess at zero on every one. The course seed's twelve draws average +0.99° and the steepest, +4.42°, arrives at 40 min. Because (6.1)

Table 6.6 Scenarios and Completion Requirements

#	Scenario	What it exercises	Required to complete
1	Standing on a stopped belt *experiment*	the single equilibrium, and basal fuel turnover	Hold the belt at zero in manual for 5 min with every state at rest, then set it to 0.8–1.0 m/s (about 2 mph) and hold it 10 min while the lags answer in order.
2	A speed step by hand *demonstration*	four lags in series, and time to exhaustion	Let both scripted manual steps land (9.4 km/h at 1 min, 11.2 km/h at 15 min) and run on until the runner goes off the back of the belt.
3	Three degrees, held *experiment*	degrees against percent against energy	Under the selector with incline feedforward off, hold the belt within 0.12 m/s of 2.18 m/s for 5 min after the first 5 min, then Reset, switch the feedforward on before Run and settle the same way again.
4	A hill arrives *experiment*	a multiplicative disturbance, and measured feedforward	With feedforward off, see lactate pass 4.0 mmol/L after the hill at 10 min, then Reset, switch the feedforward on before the hill and reach 25 min with peak lactate below 4.0 mmol/L.
5	Downhill, and the distance given away *experiment*	the asymmetry a lactate loop is worst at	Run the selector to 40 min with feedforward off, then Reset, select the predictive controller within the first minute and run the same course to 40 min.
6	Tuning the heart-rate loop into a hunt *experiment*	a hunt whose cost is not on the trend being watched	Run 15 min with the loaded tuning until command travel exceeds 12 m/s, then Reset, restore $K_{c,HR}$ to 0.9–1.3 % and $\tau_{I,HR}$ to 120–180 s before Run, and run 15 min with travel below 7.5 m/s.
7	Two runners, two binding constraints *experiment*	which constraint binds, and why it is not tuning	With the elite preset, see the heart-rate loop selected for 1 min after 10 min, then Reset, switch to the recreational preset and see the lactate loop selected for 5 min after 10 min.
8	Starting the hour under-fueled *experiment*	a constraint that is invisible until it binds	Finish the hour under the selector, then Reset, select the predictive controller within the first minute and finish the hour again.
9	A warm room and a dead fan *experiment*	a constraint with no actuator behind it	Finish the hour as loaded with peak core temperature above 39.5 °C, then Reset, restore the airflow to 0.55–0.65 and finish with the peak below 39.0 °C.
10	One hour over a rolling course *experiment*	a disturbance that is measured and not predictable	Finish the hour under the selector with feedforward off, then Reset, switch the feedforward on within the first minute and finish the same course again.
11	Overground, with a pursuer *demonstration*	a high selector over the low selector	Run under the selector until the pursuer closes the gap, then Reset, select the predictive controller within the first minute and reach 60:00 still ahead.
12	Intervals and incomplete recovery *experiment*	three stores recovering at three rates	Let the five scripted manual speed changes land, then hold the 7.9 km/h recovery for 5 min.
13	Under-fueled with a timed snack *experiment*	delayed appearance of glucose	Let the 40 g feed land at 30 min, see blood glucose rise above 5.0 mmol/L and finish the hour in automatic.
14	Belt beyond the runner's capacity *demonstration*	failure as a drift in position	Let the belt step to 15.1 km/h at 1 min in manual and run until the runner goes off the back.
15	Heart-rate monitor dropout *experiment*	a fallback cap and the return	See the fallback cap selected while the signal is lost (6 to 11 min), then run to at least 15 min with a loop selected again and the belt within 0.1 m/s of 2.6 m/s.
C	One hour, flat, for distance *challenge*	the whole economic problem under a deadline	Cover more than 9.3 km in the hour, compliant on both integrated excesses of Table 6.5, and still on the belt at 60:00, with the case's model parameters and disturbances as given.[a]

[a] With the shipped targets (86 % and 2.6 mmol/L) the hour is 8.78 km and compliant, 0.52 km short. With the suggested targets (90 % and 3.7 mmol/L) it is 9.58 km and compliant, a margin of 0.28 km, and the same when the targets are carried through Reset.

is convex the mean energy cost is 4.61 J/(kg m) against 4.15 J/(kg m) on the level, which is 11 % more for every meter and is why the same runner covers 9.05 km here against 9.58 km on the flat hour.

Table 6.7 Gradient Disturbances Under the Selector, With and Without the Measured Incline Feedforward. Scenario 4 Is Read at 25 min and Scenario 5 at 40 min, Settings Changed Before Run.

	Feedback only	With feedforward	Predictive
Scenario 4, step to 5°			
Peak lactate (mmol/L)	4.52	3.50	—
Lactate excess (mmol s/L)	29.2	0.0	—
Scenario 5, up then down			
Distance at 40 min (km)	6.04	6.25	6.50
Scenario 10, rolling course			
Distance (km)	8.80	9.05	9.38
Lactate excess (mmol s/L)	200.3	0.0	0.0
Peak lactate (mmol/L)	5.32	3.86	5.90

Scenario 5 is the same argument in the other direction and is the one runners give away. Coming onto a −2° section the sustainable speed rises from 7.30 to 12.12 km/h at the instant the deck tilts, and lactate takes four minutes to admit it. The selector reaches only 9.7 km/h over the following two and a half minutes. With the feedforward the belt is at 12.9 km/h within 30 s. The predictive controller's margin over the feedforward in Table 6.7 is 0.25 km at 40 min, which is smaller than the usual expectation, because this disturbance is measured at the moment it happens and prediction adds only the scheduling of the reserve around it.

> ✦ **DID YOU KNOW? The Longest Rolling Hour Peaks Highest**
>
> From the simulator: on the rolling course of scenario 10 the predictive controller covers the most distance, 9.38 km, reaches the highest peak lactate of the three columns, 5.90 mmol/L, and records no lactate excess. Feedback alone peaks lower, at 5.32 mmol/L, and records 200.3 mmol s/L of excess (Table 6.7).

6.7.3 Which Constraint Binds

Scenario 7 runs the elite male preset with the same controller, the same targets and the same room as the recreational preset. Both presets reach 4 mmol/L at an intensity the model resolves to a heart rate, and the two answers fall on opposite sides of the heart-rate target. The recreational preset reaches it at 160 bpm, which is 86.5 % of a 185 bpm maximum and below the 90 % target, so the lactate loop is the lower request and is selected for 96 % of everything after 10 min. The elite preset reaches it at 169 bpm, which is 91.2 % and above the target, so the heart-rate loop is selected for 81 % of the same window and the elite finishes the hour at 3.11 mmol/L with lactate it was never allowed to use. Nothing about the tuning differs between the two runs. Which constraint binds is a property of the plant, and an application that only ever showed one of the two would teach the wrong lesson about selectors.

🕮 **Pause.** Try Exercise 6.2 before reading on.

6.7.4 Detuning, and Where the Cost Lands

Scenario 6 sets the heart-rate loop ten times too hard at $K_c = 11$ % of threshold pace per bpm with its integral five times too fast at $\tau_I = 30$ s, against the 80 s of series physiological lag that Table 6.3 implies plus the 6 s measurement filter. The belt hunts over 4.7 to 8.4 km/h after the first five minutes, and the strongest spectral component of the belt-speed record between 15 and 60 min is at 338 s. Read at 40 min, the detuned loop covers 4.52 km with 43.3 m/s of command travel against 6.34 km and 9.2 m/s for the same loop returned to $K_c = 1.1$ % and $\tau_I = 150$ s before Run with the scenario's 60 s reference ramp kept, a loss of 1.82 km and a factor of 4.7 in command travel. Restoring the 300 s

ramp as well gives 6.26 km and 8.4 m/s. Five other noise seeds give 3.23 to 4.36 km detuned and 6.30 to 6.33 km restored, so the shipped seed shows the smallest loss of the six.

The instructive part is where that cost does not appear. Peak lactate over the detuned run is 1.55 mmol/L against 3.55 mmol/L for the tuned one, so the constraint everybody is watching reports the hunting controller as the safer of the two. The average of a swing that spends half its time slow is a slow average, and on this plant the penalty for a badly tuned loop is collected in the objective and in the actuator rather than in the controlled variable.

6.7.5 Constraints With and Without an Actuator

Scenario 9 places the same runner in a room at 28 °C and 60 % relative humidity with the airflow at 0.15. The runner produces the same heat and the evaporative ceiling $E_{\max}$ of (6.13) falls below it, so (6.12) has no steady state and the core temperature climbs until the pace falls far enough to balance it. The measured peak is 40.26 °C against a stop threshold of 40.5 °C, the fatigue term reaches 29 % of its ceiling, the belt finishes at 8.62 km/h and the hour is 9.21 km. Restoring the airflow to 0.60 without changing the room gives a peak of 38.76 °C and 9.53 km, so 1.5 °C of core temperature and 320 m of distance were bought by a fan and not by a controller. The predictive controller in the same room reaches 40.28 °C, 0.22 °C short of a stop, and covers 9.52 km with 5.7 mmol s/L of lactate excess. No tuning constant creates evaporative capacity, and the only control action available is to produce less heat, which means less distance.

6.7.6 Intervals, Feeding, Capacity and a Lost Signal

Scenario 12 runs two hard two-minute intervals at 11.2 km/h, separated by four easy minutes at 7.9 km/h, in manual after four minutes at 9.7 km/h. The first interval takes heart rate from 155 to 168 bpm, lactate from 2.5 to 3.9 mmol/L and the reserve from 190 to 153 m. The four easy minutes bring heart rate back to 138 bpm and lactate to 2.4 mmol/L but return only 13 m of reserve, so the second interval starts from 166 m and ends at 129 m, and ten easy minutes after it restore the reserve only to 170 m. The three quantities recover at three rates, and the one a runner feels soonest, heart rate, is the fastest of them. The reserve recovers slowly because it refills only below critical power and in proportion to the deficit in (6.7).

Scenario 13 repeats the under-fueled hour of scenario 8 with 40 g of carbohydrate eaten at thirty minutes. Blood glucose is 3.65 mmol/L when the carbohydrate goes in and 5.88 mmol/L five minutes later. Muscle glycogen at thirty-five minutes is 49.9 g against 50.8 g unfed, lower because the fed runner is running faster, and the belt holds 9.8 km/h to forty minutes. The hour is 9.43 km and compliant, against 8.90 km for the unfed hour, which also spends 23.5 mmol s/L above the lactate limit. The snack postpones the fall in sustainable pace and refills none of the muscle store.

Scenario 14 sets the belt to 15.1 km/h after a minute of standing, half again the runner's threshold pace. The belt reaches that speed about fifteen seconds after the step, and the runner keeps up while the reserve drains from 190 to 90 m by 141 s, when the capability v_{cap} of (6.18) falls below the belt speed. The runner then drifts the 0.75 m from the middle of the belt to its back edge and is off at 154.0 s, with heart rate at 141 bpm, still well behind the effort, and lactate at 6.8 mmol/L. Failure is a trajectory in position, and the reserve is the only displayed quantity that anticipates it.

In scenario 15 the chest strap stops reporting six minutes into an hour under the selector and resumes five minutes later. Meanwhile the cap holds the belt at 8.53 km/h against 9.44 km/h before the loss, and the true heart rate falls from 153 to 146 bpm while the monitor still shows 153 bpm. When the signal returns the lactate loop is selected, the belt is back at 9.36 km/h four minutes later, and the hour is 9.44 km and compliant.

6.7.7 The Full Hour, Regulator Against Optimizer

Table 6.8 reports the scorecard for the flat hour under five configurations of the same plant, read at 3600 s, and Figure 6.3 draws three of its rows.

Table 6.8 Measured Scorecard for the Flat Hour, Recreational Male Preset, Read at $t = 3600$ s, Settings Changed Before Run. The Heart-Rate IAE Accrues Over the Whole Run and Is Comparable Only Between These Columns.

	Loaded PID	Selector PID	Selector t_r=120 s	Selector no ff	Predictive MPC
HR target (%)	86	90	90	90	90
L target (mmol/L)	2.6	3.7	3.7	3.7	3.7
Distance (km)	8.784	9.584	9.677	8.676	9.947
Compliant	yes	yes	yes	yes	yes
Lactate excess (mmol s/L)	0.0	0.0	0.0	0.0	0.0
Peak lactate (mmol/L)	2.63	3.75	3.75	3.74	5.87
Peak *HR* (bpm)	155	164	164	164	182
Peak core (°C)	38.44	38.57	38.57	38.57	38.66
Reserve left (m)	190	189	185	190	36
Fuel left (miles)	12.1	11.1	11.0	12.0	10.6
Command travel (m/s)	6.3	6.5	7.0	5.2	32.9
HR IAE (bpm s)	24524	19073	23342	59603	21824
In the band (%)	36	89	89	72	96

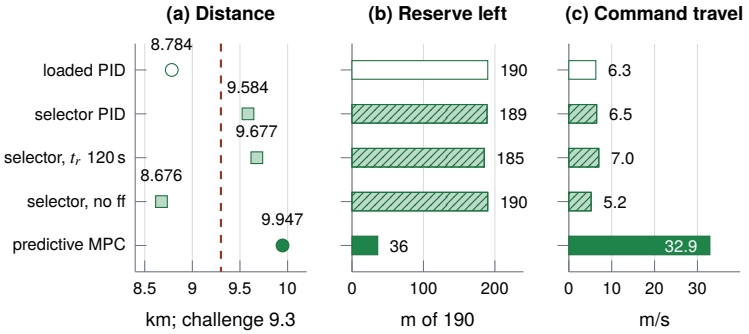


Figure 6.3 The flat hour of Table 6.8 for its five configurations, read at 3600 s: distance as dots against the challenge's 9.3 km (dashed), and the reserve left and the command travel as bars. Open marks and white bars are the loaded configuration, squares and hatched bars the three selector variants, and the solid ones the predictive controller. Notice that the predictive controller goes furthest by spending most of its reserve and by moving the belt far more than any selector.

Reading Table 6.8 across gives the expected ordering. The conservative configuration the scenario loads is comfortably inside every limit and 0.80 km short of the regulator's answer. The regulator at the suggested targets covers 9.584 km against a threshold pace that would cover 10.03 km in the hour if the runner were already at that speed at $t = 0$, and the acceleration from standing and the reference ramp account for the difference. Disabling the nominal pace feedforward of (6.23) costs 0.909 km and multiplies the heart-rate IAE by 3.1, from 19073 to 59 603 bpm s, because a proportional-integral loop starting from a stopped belt has to integrate its way up to the operating speed and the loop that gets there slowest wins the low selector. With the lactate loop's 420 s integral that is a quarter of an hour spent discovering a pace the runner's own profile already states.

The predictive controller covers 9.947 km, which is 363 m and 3.8 % more than the regulator, with every constraint respected. Five other instrument-noise seeds give gaps of 340 to 384 m. Settings made before Run and carried through Reset give identical trajectories and 32.9 m/s of command travel. Where that margin is earned is more interesting than its size, and Table 6.9 decomposes it.

Table 6.9 contradicts the intuitive account. More than half of the advantage, 210 m of 363 m, exists by 3 min, because the predictive controller reaches 14.1 km/h within 15 s and is still at 11.1 km/h at one minute while the regulator walks a 300 s heart-rate reference ramp up from rest. Between

Table 6.9 Where the Predictive Controller's 363 m Advantage Over the Selector Is Accumulated, Flat Hour, Identical Runner and Identical Targets.

Elapsed	Selector (m)	Predictive (m)	Gap (m)
1 min	42	204	162
3 min	331	541	210
10 min	1429	1690	261
30 min	4677	4959	282
55 min	8765	9025	260
60 min	9584	9947	363

Table 6.10 The Same Hours Under Three Terminal Values. *Frozen* Is the Product $(T - t_N)\, v_{\mathrm{sus}}(\mathbf{z}_N)$, *Rollout* Is Φ of (6.26), *None* Deletes Both Terminal Terms. Every Run Completes the Hour, Compliant Unless Starred (Counterfactual Controller Copy, Shipped Noise Seed, Settings Changed Before Run).

	Frozen			Rollout			None		
Hour	km	W'	tr	km	W'	tr	km	W'	tr
Flat, 300 s	9.95	46	30	9.95	36	33	9.94	38	33
Flat, 60 s	9.95	59	18	9.95	59	21	9.92*	53	25
Rolling course	9.38	38	52	9.38	42	45	9.37	31	62
Warm room	9.52*	14	68	9.52*	11	62	9.52*	21	68
Under-fueled	8.35	60	37	9.09	51	13	9.10	3	25

W' is the reserve left in meters of 190, tr is the command travel in m/s.

3 min and 55 min the predictive controller runs on average 0.06 km/h faster, at most 2.12 km/h apart, and the gap grows by only 49 m. The last five minutes add 103 m, which is the finishing allowance of (6.19) being spent. The predictive controller reaches 11.89 km/h and 5.82 mmol/L at 59.9 min while the regulator holds 9.80 km/h and 3.73 mmol/L, and the reserves end at 36 m and 189 m of 190 m respectively.

Two consequences follow. The startup component is a tuning choice rather than an architectural advantage, and shortening the reference ramp from 300 to 120 s recovers 93 m of it with no constraint violation, as the third column of Table 6.8 shows. What remains is architectural. A regulator holding a setpoint has no representation of a deadline, so it cannot decide when to stop holding, and the 153 m of reserve it carries across the finish line beyond the predictive controller's is the price of that.

6.7.8 What the Terminal Value Is Actually Worth

The usual argument for a terminal value is that without one a five-minute optimizer spends stores whose cost lands outside its horizon. That argument says nothing about whether a particular terminal value is correctly built, and it is possible to settle the question by running the same hours three ways. Table 6.10 does so, with Φ as written in (6.26), with Φ replaced by the frozen product $(T - t_N)\, v_{\mathrm{sus}}(\mathbf{z}_N)$, and with the terminal terms deleted altogether.

On four of the five hours the three forms give distances within 33 m of one another, while the distance, the reserve left and the command travel move by amounts comparable to the spread between instrument-noise seeds, and the reason is in Table 6.4. The terminal value is a function of the predicted end state through the sustainable speed, and on a level hour with full stores that speed moves 0.5 %, so the term takes very nearly the same value for every candidate plan and cannot rank them. A terminal cost that does not vary between candidates is not doing anything, whatever its magnitude.

The fifth hour is where the term is live, and the frozen form fails on it. The mechanism is a kink. The fuel factor Ψ_G in (6.5) is flat until the store falls through 15 % and slopes afterward, so $\partial v_{\mathrm{sus}} / \partial (G_m + G_l)$ is exactly zero above 54.1 g of store and 0.0103 m/s per gram below it. Multiplying

a kink by $T - t_N$, which reaches 3300 s, turns it into a cliff in J worth about 34 m per gram of predicted end-state glycogen. Above the kink a plan is charged nothing at all for burning fuel and below it the charge is abrupt, so successive replans settle on opposite sides. Between 21 and 40 min of the under-fueled hour the frozen form moves the belt between 6.2 and 10.8 km/h from one minute to the next, with 9.8, 6.4, 8.0 and 7.1 km/h at 20, 25, 30 and 35 min, which is not a pacing strategy but a controller alternating between two plans.

🕮 **Pause.** Try Exercise 6.3 before reading on.

Integrating through the kink instead of sampling one side of it removes it. The rollout in (6.26) is a distance rather than a speed times a duration, it is smooth in every state it depends on because the stores drain along it, and on the same four marks it gives 9.9, 9.8, 8.7 and 8.4 km/h. The hour recovers 746 m, from 8.35 to 9.09 km, and the command travel falls from 37 to 13 m/s, both runs compliant. The complete solve with rollout averages 0.56 ms in the local Chromium measurement over 719 advancing replans, against a 5 s replan period. These wall times depend on the host and browser.

Two conclusions follow and the second is the more useful. A terminal cost is a modeling choice with a shape, and a shape that is flat on one side of a threshold and steep on the other will be sampled rather than integrated by a receding-horizon search, which is a failure that shows up as actuator motion long before it shows up in the objective. The deeper point is in the third group of columns. With the corrected terminal value the controller is still within 31 m of having no terminal value at all on every hour tested, and on the under-fueled hour the version without one is the 6 m longer. The term is not what makes this predictive controller beat the regulator. What does that is the constraint handling and the finishing allowance of (6.19), and the honest description of the terminal value here is that its job is to avoid doing harm.

6.8 Technology in Practice

The heart-rate monitor, the lactate estimate and the belt motor of Figure 6.1 stand for the equipment of an exercise laboratory. Figure 6.5 draws the selector loop of Section 6.5 with that equipment in place of the schematic's symbols.

What Measures It

Heart rate comes from a chest strap that records the heart's electrical signal and reports a rate over a wireless link. In one laboratory heart-rate loop the rate arrived once a second and the controller used the average of five readings every 5 s [8]. The chapter's measurement in Equation (6.21), with 0.8 bpm of colored noise and a 6 s filter, stands for that strap and its averaging, and the held reading of scenario 15 stands for a strap that stops reporting. Lactate has no such instrument: a threshold test draws blood at the fingertip or the earlobe [23]. The lactate signal the chapter's loop reads every second is therefore an estimate, and the chapter labels it so.

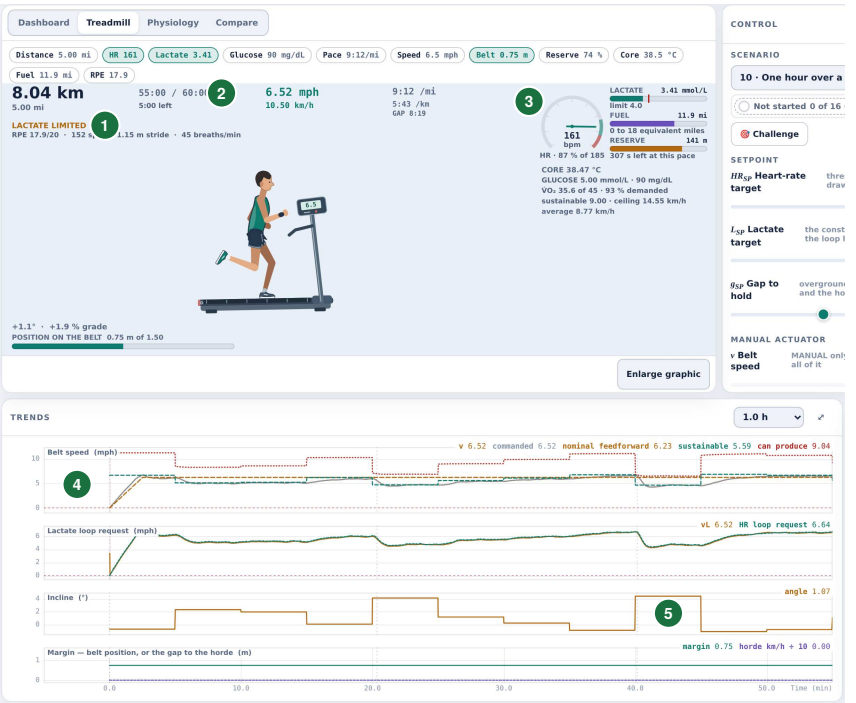


Figure 6.4 Scenario 10 at $t = 55$ min, the rolling course under the selector on feedback alone, cropped to the treadmill view and the trends. The deck is tilted to +1.07° and the runner has covered 8.04 km with lactate at 3.41 mmol/L. The incline strip records the five-minute redraws over the whole run. The +4.42° block that lands at 40 min takes the sustainable pace down, drives blood lactate above its 4.0 mmol/L constraint within 9 s and pulls the lactate loop's request and the belt down with it, the belt to 6.83 km/h at 41:13, and that block adds 169 of the 200 mmol s/L of lactate excess the run records. The belt-speed strip carries the commanded speed, the nominal feedforward of (6.23), the sustainable pace and the capability v_{cap} alongside the belt itself, and the scene names the selected constraint in words above the machine. Callouts: (1) the selected constraint named in words, (2) distance and elapsed time, (3) the heart-rate gauge with the lactate, fuel and reserve bars, (4) the belt-speed strip, (5) the +4.42° block on the incline strip.

TECHNOLOGY **Sensor: The Chest-Strap Heart-Rate Monitor**

A chest strap holds electrodes against the skin and records the heart's electrical signal, from which it reports the heart rate and the intervals between beats over a wireless link. In a 2025 systematic review the most studied straps correlated strongly with electrocardiography at rest and in light-to-moderate activity, and chest straps were noted for better accuracy than the more common wrist-worn devices. The limitations reported were motion artifacts, poor strap placement, sweating and a degrading contact between skin and electrode, and none of the devices reviewed was approved for clinical use [24]. A clinical cardiac monitor that measures heart rate from an electrocardiograph's signal, and may alarm outside preset limits, is a Class II device in United States regulation [25].

What Moves It

The final element is the belt drive, which brings the belt to the speed a computer commands [26]. The chapter models that inner loop in Equation (6.18) as a first-order motor with a 0.6 s lag and rate limits of 0.35 and 0.50 m/s^2, and its automatic modes limit the command itself to 1.5 times the belt's rate limits. A runner who cannot hold the belt speed drifts back along the deck, which is the position state of the chapter and the failure that scenario 14 shows.

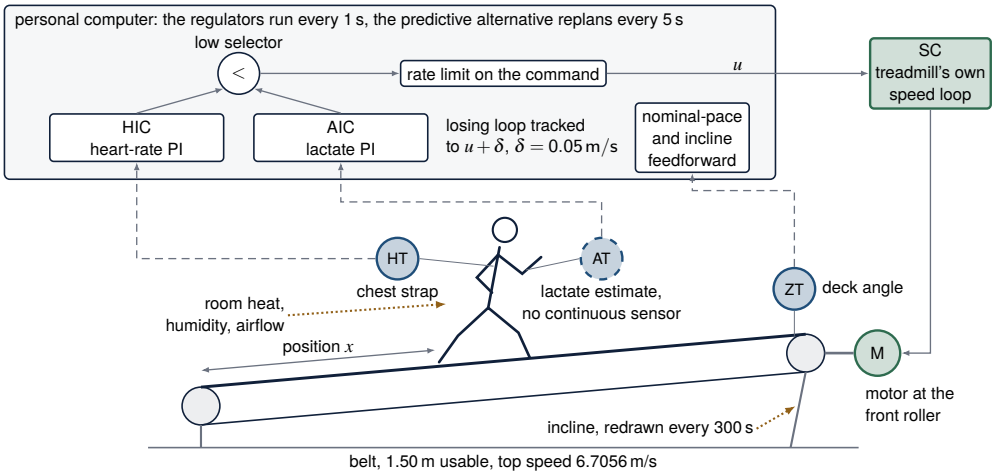


Figure 6.5 The selector loop of Section 6.5 with laboratory equipment in place of the schematic's symbols. A chest strap (HT) reports heart rate and the deck angle (ZT) is measured, while the lactate signal (AT, dashed) is an estimate, because the chapter implies no continuous blood-lactate sensor. Two PI loops (HIC, AIC) meet in a low selector on a personal computer, and its rate-limited command u goes to the treadmill's own speed loop (SC) and motor (M). Notice that one final element serves both loops, and that the runner's position x on the belt, the state that ends a run, is read by no instrument in this loop.

TECHNOLOGY **Final Element: The Treadmill Belt Drive**

A treadmill in a control study takes its speed from a computer, and its own drive holds the belt there. In an adaptive running platform the computed speed went to the platform's own microcontroller, which set it directly, many times a second, and the platform accelerated and stopped at a constant 1 m/s^2 [26]. A predictive heart-rate controller kept both the speed and the acceleration of its treadmill within a safe range for vulnerable exercisers [10]. United States regulation lists a powered treadmill intended for medical purposes among powered exercise equipment, a Class I device under general controls [27].

Where the Controller Runs

In the laboratory the heart-rate controller ran as real-time software on a personal computer beside the treadmill and sent its speed commands over a serial link every 5 s [8]. The chapter's regulators run once per simulated second and its optimizer replans every 5 s. The two proportional-integral loops, the low selector with its external-reset band, the two feedforward terms and the fallback cap of Section 6.5 form one supervisory layer above the drive's own speed loop.

IN PRACTICE **Running a Heart-Rate Loop in the Laboratory**

A laboratory test of a heart-rate loop is a protocol as much as a controller. In one study each test began with 10 min of warm-up under the controller, then 10 min of rest, then a 35 min measurement, and the target heart rate was set from the age-predicted maximum. The tracking error was scored over the 30 minutes from 295 to 2095 s. Two participants had measurement artifacts in their final five minutes, and their scoring stopped five minutes early. Beside the tracking error, the study scored the controller's effort as the average power of the changes in the speed command [8], the concern that command travel answers in Table 6.5.

What Is Modernizing It

Three developments bear on this loop. The first is predictive control on models identified from the person: a predictive heart-rate controller ran on models estimated with a short protocol and kept the

treadmill's speed and acceleration within a safe range [10]. The second is a treadmill that follows the runner. An adaptive running platform reads the person's position on the belt from trackers or a camera and sets the belt speed from it, and a position error up to 5 cm, 30 Hz updates and 100 ms of total latency were judged acceptable [26]. That loop is closed on the position x that the chapter uses only as its failure state. The third is lactate read from the skin. A wearable sensor on the forehead monitored sweat lactate continuously during arm crank exercise in 27 healthy young adults, and the threshold read from it correlated strongly with the first ventilatory threshold [28]. A sensor of that kind that held for every runner would replace the chapter's lactate estimate with a measurement.

6.9 Limitations

The model is not a physiological instrument and cannot be used as one. The coefficients of (6.5), (6.10), (6.11) and (6.13) are teaching calibrations chosen so that four presets behave plausibly, and they are not fitted to any individual or population. The relations they are built on are published, and the parameterizations that carry them here are not. Nothing in the application is validated for prescribing training, assessing fitness or guiding anybody's running.

Critical power and the 4 mmol/L intensity are equal in every preset, which removes a distinction the literature maintains, so the difference between the two constructs can be studied only by editing one of them. Lactate has no transport delay beyond its clearance time constant, so the application understates the measurement problem rather than overstating it. The thermal model has one node and no dehydration, so a hot hour degrades more gently here than in practice. Feeding is carbohydrate alone, and nothing refills muscle glycogen during the run.

The predictive controller finds an approximate local solution with a bounded coordinate search over four blocked moves and a rolled-out terminal value. It is not a global optimum and no claim of optimality is made, and the comparison in Table 6.9 is a comparison between two implemented controllers rather than between an architecture and a bound. The teaching switch that hands the predictive controller the hidden incline schedule is an advantage no controller has, and any comparison that uses it is labeled on screen and in the exported record. The determinism is unrealistic in its own right, since a person running the same hour twice does not produce the same hour twice. Section 6.3.9 names the point at which each modeling assumption stops holding.

> **OPEN CHALLENGE** **A Lactate Reading Without a Blood Sample**
>
> The chapter labels its lactate signal an estimate and implies no continuous blood-lactate sensor. In practice most tests of the anaerobic threshold still draw blood at the fingertip or earlobe, which is costly and time-consuming. A 2025 study of 55 subjects on a cycle ergometer read thresholds from sweat electrolytes instead. They agreed with the blood lactate threshold in the high and middle fitness groups but may not predict it in the low group, and the authors called for further research [23]. Progress would be a reading taken from sweat or skin that holds for every fitness level, which is the gap that study left open.

6.10 Discussion

The structure of this chapter is not particular to running. Any plant with a stored resource, a fixed deadline and an objective that grows with the thing the constraints forbid has the same shape, and the list includes a battery-electric vehicle on a long climb, a peaking generator working against a fuel contract, a batch reactor with a fixed cycle time and a spacecraft with a propellant budget. Volume 9, Chapter 12 spends a train's spare seconds between stations against its energy meter on the same kind of deadline. In each of them the value of the store at the deadline is nothing like its value halfway through, which is the property that a regulator cannot represent and an economic formulation can [2]. Pacing models in sport reached the same conclusion from the other direction, by writing the event as an energy flow with a finite anaerobic store and solving for the schedule [3]. Putting both on one actuator lets the 363 m in Table 6.9 be split into the part that is a tuning choice and the part that is architectural.

Tables 6.4 and 6.10 carry one caution between them. What makes scheduling a store valuable is a constraint that moves, and on the level hour the sustainable speed moves 0.5 %, which is why the flat hour moves by less than 12 m whether the terminal value is computed carefully, computed carelessly or deleted. Only in the warm room and the under-fueled hour, where the constraint falls 11.4 % and 20.2 %, does the forecast of a store have any work to do, and the under-fueled hour is the only one on which the careless form does damage. Establishing which regime a plant is in, before building the controller that assumes one of them, decides whether the terminal cost is worth writing and whether its shape matters.

Two further methods fit the same equations without changing them. The fourteen-state system with explicit constraints transfers directly to a nonlinear programming formulation in APMonitor [29] or the Gekko Optimization Suite [30], where the blocked search of (6.25) would be replaced by a gradient-based solve over a full discretization and the rollout of (6.26) by a cost-to-go the solver differentiates. A moving horizon estimator built on the same equations could recover ϕ, the one quantity the application keeps off every display, from the heart rate and the pace still available, which turns the heat-against-fuel diagnosis of Table 6.4 into a soft sensor instead of a reading of two trends. The heart-rate identification literature supplies the reduced models a practical implementation would use in place of the full plant [4, 5, 10].

6.11 Conclusion

This chapter presents one hour of running as a laboratory plant in which a single actuator serves several constrained outputs and the objective is bounded by a deadline. The simulator integrates fourteen states, covering the mechanical belt and the runner’s position on it, oxygen uptake, a finite anaerobic reserve, heart rate, blood lactate, accumulated fatigue, core temperature, respiration, three carbohydrate stores and a gut compartment for carbohydrate eaten during the run, with the energy cost of gradient taken from published measurement and the sustainable power written as a critical-power construct that falls with fatigue and fuel. The four constrained outputs answer a belt step with time constants of 40, 96, 257 and 475 s. Two proportional-integral loops sharing the actuator through a low selector with a 0.05 m/s external-reset band cover 9.584 km in the hour with every constraint respected, while a receding-horizon controller with a 300 s horizon and a terminal value covers 9.947 km and finishes with 36 m of its 190 m reserve against the regulator’s 189 m. Of that 363 m advantage, 210 m exists by three minutes and is recoverable by shortening the regulator’s reference ramp, and 103 m is taken in the last five minutes when the ceilings open. A measured incline feedforward removes the whole of the 200.3 mmol s/L of lactate excess that twelve hills cost feedback alone, and detuning the heart-rate loop costs 1.82 km of the first 40 min and 4.7 times the command travel while lowering the peak lactate, which is the reason the scorecard reports competing costs rather than one number. The terminal value is measured rather than assumed. Written as the end-state sustainable speed held for the remaining time it inherits a kink from the fuel factor, the search alternates across it, and the under-fueled hour loses 746 m and raises its command travel from 13 to 37 m/s. Written as a rollout of the base policy it does not, and it is then within 31 m of having no terminal value at all on every hour tested. The later cases show the reserve recovering far more slowly than heart rate between intervals, a snack at half time worth 535 m to an under-fueled runner without refilling the muscle store, a belt beyond capacity failing as a drift in position 94 s after the step, and a lost heart-rate signal handled by a conservative cap. A fuller model would add a skin compartment and dehydration in the thermal balance, presets that separate critical power from the lactate marker, a gradient-based solve in place of the blocked coordinate search, and estimation of the hidden fatigue state from the measurements the runner already has. A runner on a belt is an accessible plant for learning to hold whichever constraint binds and to work out when to stop holding it.

KEY POINTS

- One belt speed has to respect several limits of the runner's body, and which limit is reached first depends on the runner, not on the tuning.
- A low selector passes the lower of the two requests. Its losing loop is tracked to a band above the applied speed, because tracking it to the applied speed itself deadlocks the selector.
- A measured incline feedforward removes the whole 200.3 mmol s/L of lactate excess that twelve hills cost feedback alone.
- The predictive controller covers 9.947 km against 9.584 km. Of that 363 m, 210 m exists by three minutes and is a tuning choice.

Where else this applies. One actuator shared among limits by a selector returns in Volume 4, Chapter 4 and Volume 4, Chapter 3, and slack spent against a fixed deadline in Volume 9, Chapter 12.

References

[1] K. J. Åström and T. Hägglund, *Advanced PID Control*. Research Triangle Park, NC: ISA – The Instrumentation, Systems, and Automation Society, 2006.

[2] M. Ellis, H. Durand, and P. D. Christofides, "A tutorial review of economic model predictive control methods," *Journal of Process Control*, vol. 24, no. 8, pp. 1156–1178, 2014.

[3] J. J. de Koning, M. F. Bobbert, and C. Foster, "Determination of optimal pacing strategy in track cycling with an energy flow model," *Journal of Science and Medicine in Sport*, vol. 2, no. 3, pp. 266–277, 1999.

[4] S. W. Su, L. Wang, B. G. Celler, A. V. Savkin, and Y. Guo, "Identification and control for heart rate regulation during treadmill exercise," *IEEE Transactions on Biomedical Engineering*, vol. 54, no. 7, pp. 1238–1246, 2007.

[5] T. M. Cheng, A. V. Savkin, B. G. Celler, S. W. Su, and L. Wang, "Nonlinear modeling and control of human heart rate response during exercise with various work load intensities," *IEEE Transactions on Biomedical Engineering*, vol. 55, no. 11, pp. 2499–2508, 2008.

[6] K. J. Hunt and S. E. Fankhauser, "Heart rate control during treadmill exercise using input-sensitivity shaping for disturbance rejection of very-low-frequency heart rate variability," *Biomedical Signal Processing and Control*, vol. 30, pp. 31–42, 2016.

[7] K. J. Hunt, B. Ajayi, H. Gollee, and L. Jamieson, "Feedback control of oxygen uptake during treadmill exercise," *IEEE Transactions on Control Systems Technology*, vol. 16, no. 4, pp. 624–635, 2008.

[8] H. Wang and K. J. Hunt, "Feedback control of heart rate during treadmill exercise based on a two-phase response model," *PLOS ONE*, vol. 18, no. 10, p. e0292310, 2023.

[9] R. A. Bruce, F. Kusumi, and D. Hosmer, "Maximal oxygen intake and nomographic assessment of functional aerobic impairment in cardiovascular disease," *American Heart Journal*, vol. 85, no. 4, pp. 546–562, 1973.

[10] L. Wang, Y. Yang, and S. Su, "Nonparametric modelling based model predictive control for human heart rate regulation during treadmill exercise," in *2021 43rd Annual International Conference of the IEEE Engineering in Medicine & Biology Society (EMBC)*. IEEE, 2021, pp. 148–151.

[11] A. E. Minetti, C. Moia, G. S. Roi, D. Susta, and G. Ferretti, "Energy cost of walking and running at extreme uphill and downhill slopes," *Journal of Applied Physiology*, vol. 93, no. 3, pp. 1039–1046, 2002.

[12] H. Monod and J. Scherrer, "The work capacity of a synergic muscular group," *Ergonomics*, vol. 8, no. 3, pp. 329–338, 1965.

[13] A. M. Jones, A. Vanhatalo, M. Burnley, R. H. Morton, and D. C. Poole, "Critical power: Implications for determination of $\dot{V}O_2$max and exercise tolerance," *Medicine & Science in Sports & Exercise*, vol. 42, no. 10, pp. 1876–1890, 2010.

[14] D. P. Swain and B. C. Leutholtz, "Heart rate reserve is equivalent to $\%\dot{V}O_2$ reserve, not to $\%\dot{V}O_2$max," *Medicine & Science in Sports & Exercise*, vol. 29, no. 3, pp. 410–414, 1997.

[15] H. Tanaka, K. D. Monahan, and D. R. Seals, "Age-predicted maximal heart rate revisited," *Journal of the American College of Cardiology*, vol. 37, no. 1, pp. 153–156, 2001.

[16] G. A. Brooks, "The science and translation of lactate shuttle theory," *Cell Metabolism*, vol. 27, no. 4, pp. 757–785, 2018.

[17] H. Heck, A. Mader, G. Hess, S. Mücke, R. Müller, and W. Hollmann, "Justification of the 4-mmol/l lactate threshold," *International Journal of Sports Medicine*, vol. 6, no. 3, pp. 117–130, 1985.

[18] P. L. Valenzuela, L. B. Alejo, A. Montalvo-Pérez, J. Gil-Cabrera, E. Talavera, A. Lucia, and D. Barranco-Gil, "Relationship between critical power and different lactate threshold markers in recreational cyclists," *Frontiers in Physiology*, vol. 12, p. 676484, 2021.

[19] H. Heck and H. Wackerhage, "The origin of the maximal lactate steady state (MLSS)," *BMC Sports Science, Medicine and Rehabilitation*, vol. 16, p. 36, 2024.

[20] J. González-Alonso, C. Teller, S. L. Andersen, F. B. Jensen, T. Hyldig, and B. Nielsen, "Influence of body temperature on the development of fatigue during prolonged exercise in the heat," *Journal of Applied Physiology*, vol. 86, no. 3, pp. 1032–1039, 1999.

[21] J. A. Romijn, E. F. Coyle, L. S. Sidossis, A. Gastaldelli, J. F. Horowitz, E. Endert, and R. R. Wolfe, "Regulation of endogenous fat and carbohydrate metabolism in relation to exercise intensity and duration," *American Journal of Physiology-Endocrinology and Metabolism*, vol. 265, no. 3, pp. E380–E391, 1993.

[22] E. F. Coyle, A. R. Coggan, M. K. Hemmert, and J. L. Ivy, "Muscle glycogen utilization during prolonged strenuous exercise when fed carbohydrate," *Journal of Applied Physiology*, vol. 61, no. 1, pp. 165–172, 1986.

[23] Z. Luo, M. Cao, L. Yan, and J. Wang, "Feasibility study of a novel wearable sweat sensor for anaerobic threshold determination," *Scientific Reports*, vol. 15, p. 30467, 2025.

[24] A. Machado, D. F. Ferreira, S. Ferreira, N. Almeida-Antunes, P. Carvalho, P. Melo, N. Rocha, and M. A. Rodrigues, "A systematic review of chest-worn sensors in cardiac assessment: Technologies, advantages, and limitations," *Sensors*, vol. 25, no. 19, p. 6049, 2025.

[25] U.S. Food and Drug Administration, "Cardiac monitor (including cardiotachometer and rate alarm)," Code of Federal Regulations, Title 21, Section 870.2300, revised as of April 1, 2024, U.S. Government Publishing Office, govinfo.gov, 2024.

[26] A. Obukhov and A. Volkov, "Development of control algorithms for an adaptive running platform for a musculoskeletal rehabilitation system," *Sensors*, vol. 25, no. 21, p. 6667, 2025.

[27] U.S. Food and Drug Administration, "Powered exercise equipment," Code of Federal Regulations, Title 21, Section 890.5380, revised as of April 1, 2024, U.S. Government Publishing Office, govinfo.gov, 2024.

[28] T. Sawada, H. Okawara, K. Minami, S. Narushima, A. Shiratori, Y. Katsumata, M. Nakamura, T. Nagura, and D. Nakashima, "Validation of aerobic threshold assessment using a sweat lactate sensor during arm crank exercise by healthy adults," *Physiological Reports*, vol. 14, no. 13, p. e71002, 2026.

[29] J. D. Hedengren, R. Asgharzadeh Shishavan, K. M. Powell, and T. F. Edgar, "Nonlinear modeling, estimation and predictive control in APMonitor," *Computers & Chemical Engineering*, vol. 70, pp. 133–148, 2014.

[30] L. D. R. Beal, D. C. Hill, R. A. Martin, and J. D. Hedengren, "GEKKO optimization suite," *Processes*, vol. 6, no. 8, p. 106, 2018.

6.12 Exercises

6.1 ○ Concept check · 2 min

Suppose the band δ of Equation (6.24) is set to zero, so that the loop that lost the comparison is tracked to exactly the applied speed u. What does the treadmill do?

(a) The losing integrator winds to the top of the belt's range, so the hand-over comes late

(b) The belt follows the higher of the two requests, so the binding limit is exceeded

(c) The selection settles on whichever loop has the shorter integral time

(d) The loops trade the selection every scan, and the belt stops moving toward either target

6.2 ○ Concept check · 1 min

Scenario 7 runs the elite male preset with the same tuning, targets and room as the recreational preset. Which loop holds the belt for most of the run after 10 min, and why?

(a) The heart-rate loop, because this runner reaches 4 mmol/L above the 90 % heart-rate target

(b) The lactate loop, because its gain is the larger of the two in percent of threshold pace

(c) The heart-rate loop, because this runner's heart rate lags the belt less than lactate does

(d) The lactate loop, because this runner's anaerobic reserve is larger than the other's

6.3 Concept check · 2 min

With the frozen terminal value $(T - t_N)\, v_{\text{sus}}(\mathbf{z}_N)$, the under-fueled hour moves the belt between 6.2 and 10.8 km/h from one minute to the next. What causes the alternation?

(a) The move penalty λ is too small to damp the plan between replans

(b) The fuel factor has a kink at 15 % of the store, which the remaining time turns into a cliff in J

(c) The lactate estimate's noise changes which plans are feasible at every replan

(d) The horizon cannot see the finishing window until the last five minutes

6.4 Theory and calculation · 6 min

Take the recreational male preset of Table 6.2: $m = 75\,\text{kg}$, $\dot{V}_{O_2,\max} = 45\,\text{mL/(kg min)}$, $C_r = 4.15\,\text{J/(kg m)}$, $f_{CP} = 0.80$, $HR_{\text{rest}} = 60\,\text{bpm}$ and age 35, with $\varepsilon_{O_2} = 20.9\,\text{J/mL}$.

(a) Compute P_{rest} and $P_{\max}$ from Equations (6.2) and (6.3). *Checkpoint:* 91.4 W and 1175.6 W.

(b) Evaluate the static gain of Equation (6.20). *Checkpoint:* 35.9 bpm per m/s.

(c) On the level with full stores and no fatigue, the threshold pace v_{nom} satisfies $mC_r\, v_{\text{nom}} = CP - P_{\text{rest}}$. Show that $v_{\text{nom}}\, \partial HR/\partial v$ equals $f_{CP}\,(HR_{\max} - HR_{\text{rest}})$. Then compute the loop gain $K_c\, v_{\text{nom}}\, \partial HR/\partial v$ of the heart-rate loop at $K_c = 1.1\,\%$ of threshold pace per bpm, for this preset and for the elite male preset ($f_{CP} = 0.87$, same age and resting heart rate). Give two decimals. *Checkpoint:* 1.10 and 1.20.

6.5 Theory and calculation · 6 min

The recreational male preset on the level has $CP = 958.8\,\text{W}$, $P_{\text{rest}} = 91.4\,\text{W}$, a reserve $W'_{\max} = mC_r D'$ with $D' = 190\,\text{m}$, and $\mu = 1.35$. Treat ϕ as zero and the belt as being at speed from the start.

(a) At a constant 11.2 km/h, compute the demand P of Equation (6.2), the excess $P - CP$ and the time to empty a full reserve. *Checkpoint:* 1059.8 W, 101.0 W, 586 s.

(b) The legs fall short of the belt when P_{av} of Equation (6.6) drops below P. Find the fraction $W'/W'_{\max}$ at which that happens at 11.2 km/h, and the time to reach it. Compare with the failure at 577 s that the chapter reports. *Checkpoint:* 0.0118 and 579 s.

(c) Repeat (b) for the 15.1 km/h belt of scenario 14 and give the reserve in meters at which the runner loses the belt. *Checkpoint:* 88 m.

6.6 Theory and calculation · 5 min

The frozen terminal value multiplies the sustainable speed of the predicted end state by the time left, $T - t_N$. Use the recreational male preset on the level with $\phi = 0$, a blood glucose above 4 mmol/L and the starting store $G_{m,0} + G_{l,0} = 361\,\text{g}$.

(a) Below what store, in grams, does Ψ_G of Equation (6.5) begin to fall? *Checkpoint:* 54.1 g.

(b) Below that store, compute $\partial v_{\text{sus}}/\partial(G_m + G_l)$, with $v_{\text{sus}} = (CP - P_{\text{rest}})/(mC_r)$. *Checkpoint:* 0.0103 m/s per gram.

(c) Multiply by the largest $T - t_N$, 3300 s, and explain what a bounded coordinate search does with an objective shaped this way. *Checkpoint:* about 34 m per gram.

6.7 App challenge · 8 min

Load scenario 4, A hill arrives, and keep the seed and every setting it loads (Auto · PID, recreational male, incline feedforward off). The deck tilts to 5° at 10 min. Set the units to SI (km/h and kilometers) and run to 25:00 at speed 20.

(a) Read the Lactate excess tile and the peak lactate, given in the tile's hover note and in the last row of column `metric_peak_lac` of the exported CSV file. Give the excess within 1 mmol s/L and the peak within 0.02 mmol/L. *Checkpoint:* 29 mmol s/L and 4.52 mmol/L.

(b) Reload the scenario, set Incline feedforward to On under Advanced before Run, and repeat the run and the readings. *Checkpoint:* 0 mmol s/L and 3.50 mmol/L.

(c) With $C = 4.15\,\mathrm{J/(kg\,m)}$ on the level and 6.49 J/(kg m) at 5°, compute the speed that Equation (6.23) sets on the hill from the threshold pace of 2.787 m/s. Check it against column `speed_ff_mps` in the row at 20 min of the second file, within 0.01 m/s. *Checkpoint:* 1.78 m/s.

6.8 App challenge · 8 min

Load scenario 6, Tuning the heart-rate loop into a hunt, and keep the seed and every setting it loads ($K_c = 11\,\%$, $\tau_I = 30\,\mathrm{s}$, a 60 s reference ramp, monitor noise 2 bpm). Set the units to SI (km/h and kilometers), run to 40:00 at speed 60 and export the CSV file.

(a) Read the Distance tile, the peak lactate (hover note of the Lactate excess tile) and the command travel, the last row of column `metric_mv_travel` in m/s. *Checkpoint:* 4.52 km, 1.55 mmol/L, 43.3 m/s.

(b) Reload the scenario, set Heart-rate loop gain to 1.10 % and Heart-rate integral time to 150 s before Run, leave the ramp at 60 s, and repeat. *Checkpoint:* 6.34 km, 3.55 mmol/L, 9.2 m/s.

(c) Compute the distance lost to the hunt and the ratio of the command travels. Which of the three readings would mislead an operator watching only the constrained variable? *Checkpoint:* 1.82 km and 4.7.

Tolerances: 0.02 km, 0.02 mmol/L and 0.5 m/s.

6.9 App challenge · 9 min

Load the challenge, One hour, flat, for distance, and keep its seed. Before Run set Heart-rate target to 90 % and Lactate target to 3.7 mmol/L, keep Auto · PID, set the units to SI (km/h and kilometers), run the hour at speed 60 and export the CSV file. Skip any lines that start with #. The file's time column is in hours.

(a) Read the last row of column `dist_km` and the Reserve left tile. *Checkpoint:* 9.584 km and 189 m.

(b) Reload, set the same two targets and Auto · MPC before Run, and repeat. *Checkpoint:* 9.947 km and 36 m.

(c) In both files read `dist_km` in the row at 0.05 h. What share of the final gap exists by three minutes? *Checkpoint:* 0.331 km and 0.541 km, a gap of 210 m, 58 % of 363 m.

(d) Using Table 6.8, say which part of the gap is a tuning choice and which is architectural. *Checkpoint:* The start is a tuning choice, and the finish is architectural.

Tolerances: 0.005 km on distances and 1 m on the reserve.

6.10 Application area · 3 min

A clinic wants to use the application's heart-rate band, 83 to 92 % of 220 − age, for a 60-year-old client.

(a) Compute the estimated maximum and the band. *Checkpoint:* 160 bpm, 133 to 147 bpm.

(b) The estimator carries a population standard deviation near 11 bpm. How far can the band sit from the client's true band, and what would you set in the application, from what measurement? *Checkpoint:* About ten beats. Set Δ_{HR} from a measured maximum.

6.11 Application area · 4 min

Scenario 9 puts the recreational runner in a room at 28 °C and 60 % relative humidity with the airflow at 0.15.

(a) From Equation (6.13) with $E_0 = 1820\,\mathrm{W}$ and $\Phi = 1$, compute the evaporative ceiling $E_{\max}$ at airflow 0.15 and at 0.60. *Checkpoint:* 620 W and 909 W.

(b) At threshold pace the runner must shed about $CP - P_{\mathrm{rest}}$, or 867 W, above the resting turnover. At which airflow can the heat balance reach a steady state, and what does that say about the actuator for core temperature? *Checkpoint:* Only at 0.60. The actuator is the fan, not the belt loops.

6.12 Review the agent · 5 min

An AI assistant reads Table 6.8 and writes: "The predictive controller covers 363 m more than the selector because its terminal value prices the reserve beyond the 300 s horizon. To close the gap, give the regulator the same terminal value, and to widen it, double the weight of the predictive controller's terminal terms." Find the flaw, and say how you would check the claim in the chapter and in the app.