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CHAPTER 15

A Juggling Robot Arm Whose Paddle Must Be Slowing at Impact

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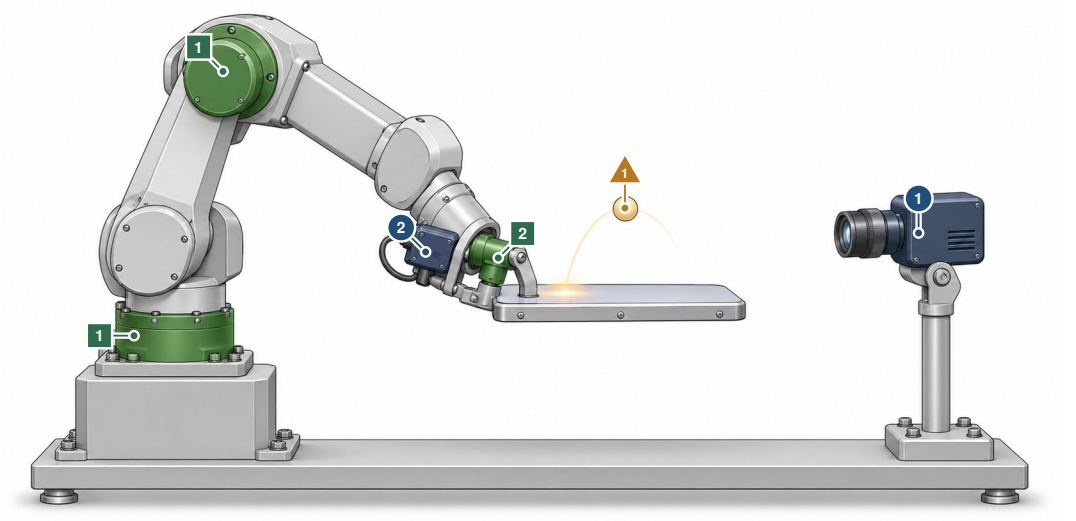
The app for this chapter
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CHAPTER 15

A Juggling Robot Arm Whose Paddle Must Be Slowing at Impact



The controller moves 1 the arm's joint drives, which move the paddle up and down
2 the wrist drive that tilts the paddle face

The controller reads 1 the camera that sees the ball, late 2 the impact switch at the paddle mount

What disturbs it 1 a different ball, whose restitution changes

The arm meets the ball while the paddle is rising but already slowing, so notice that the camera on its stand is not needed to keep the ball up: the blind stroke reads no sensor, and only the mirror and tilt laws read the camera. Illustration, not to scale.

IN PLAIN TERMS Keeping a Ball Bouncing on a Paddle

The main idea. A robot arm holds a paddle under a rubber ball and keeps it bouncing to the same height. Each bounce is a short flight between two hits, so the paddle is judged once a bounce, not all the time.

The challenge. The paddle must meet the ball while it moves up but is already slowing. Then a ball that comes back late gets a softer push and falls back into step, with no camera at all. A camera whose pictures come too late cannot make up the lost time.

Why it matters. Walking robots and many machines that repeat one motion are kept steady in the same way, one cycle at a time.

THREE WAYS TO READ THIS CHAPTER

Explore	12 min	the first page, the history in Section 15.2, the fact in Section 15.4, the two try-its and the fact in Section 15.6, and the key points
Study	50 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	15 min	the first page, the results (Section 15.6), Technology in Practice (Section 15.7), Limitations (Section 15.8) and the open challenge

The control question A ball kept bouncing on a paddle is a hybrid system, and what the juggler holds is a periodic orbit rather than a height. When does a paddle stroke keep the ball up with no sensor at all, and what can feedback through an impact switch or a late camera add to it?

Learning objectives

- Derive the paddle speed at impact that holds a period-1 bounce, and the linearized impact map whose determinant is e^2.
- Compute the window of paddle accelerations at impact in which the orbit is stable, and place a sinusoidal stroke in it.
- Measure a ball's restitution from its bounces on a still paddle, and explain why infinitely many impacts end in finite time.
- Compare the open-loop sinusoid, the once-per-bounce PI law and the mirror law through a change of ball.
- Decide when a sensor's latency leaves a feedback law no orbit to hold.

Prerequisites Sections 1.4.7 (hybrid automata and Zeno solutions), 1.4.9 (maps from one cycle to the next), 1.6.1 (the engine and its seed) and 1.6.4 (scenarios and the challenge). Chapter 12 shows another plant whose switching is part of its dynamics.

The app https://apmonitor.com/apps/paddle-juggler/

See also Volume 7, Chapter 9 for a plant that is self-stable only inside a band of an operating variable. Volume 20, Chapter 11 for the same six-axis arm striking a ball, and Volume 20, Chapter 10 for an arm acting on late camera reports.

15.1 Introduction

Dropped from 50 cm onto a paddle that does not move, a 40 g rubber ball rises to 33, 22, 14 and 9 cm and lies still on the face after 29 impacts in 2.98 s. To send it back to 50 cm every time, the paddle has to meet it while moving up at 0.348 m/s, the speed that replaces what a restitution of 0.80 takes away. A stroke of 6.3 cm each way at 1.655 Hz does that with no sensor at all and holds the apex at 0.502 m with a spread of 1.6 cm over a 30 s run, because the paddle is decelerating at 5.9 m/s^2 when the two meet. A stroke of 3 cm at the same frequency lets the ball die on the paddle, and raised to 11 cm while it runs the stroke throws the ball anywhere from 0.18 to 0.98 m. Each of these runs was made on the simulator of this chapter, a ball juggled by the six-axis arm of a simulated table-tennis robot, and this chapter explains why they end so differently.

The juggler unsettles three beliefs that a first control course tends to leave in place. The first is that the object of control is a setpoint. Here the target is a whole motion repeated every 0.6 s, and its stability is decided between one impact and the next, by a map whose eigenvalues no step response shows. The second is that stability needs a measurement. A paddle that is slowing down when it meets the ball corrects a late bounce with a smaller push and an early one with a larger push, so the orbit pulls itself back with nothing measuring the ball, and with the camera switched off the same stroke carries the ball through a change to a restitution of 0.72 without a drop at six seeds of six. The third is that more gain makes up for a slow sensor. Through a camera 120 ms late, every mirror-law gain from 0.25 to 2.5 lets a 0.20 m bounce die in under 2.4 s, while a blind 3 cm sinusoid at 2.5 Hz keeps it up.

A ball on a paddle belongs to a broad class, mechanical systems whose periodic motions contain impacts, studied as systems with jumps whose solutions mix flows and resets [1, 2], the hybrid automata of Section 1.4.7. A walking machine strikes the ground once a step and its gait is a periodic orbit read on a return map [3, 4], and a robot that throws, bats or rolls an object it never grasps is a manipulator of the same kind [5]. The bouncing ball on a vibrating table is the simplest member, and its period doublings and chaos are a standard example of nonlinear dynamics [6–8]. A press stroke, a cam-driven packaging line and a batch corrected after every run, as in Chapter 9, share its structure, a cycle whose stability is a property of the cycle and a controller whose sample is an event.

However, the robots that juggle are each described around one platform, a linear actuator, a planar juggler or a vision-guided arm [9–12], and the analysis of the impact map is usually done for an ideal paddle that follows its waveform exactly [6, 13]. A student cannot see, on one machine, the map's eigenvalues beside an arm whose drives have limits, a camera that is late and a ball that changes. This chapter shows how much the stability window, the ball's restitution, a sensor's latency and a once-per-bounce controller each decide when a full-sized arm keeps a ball on an orbit, using a browser simulator that logs every bounce with the paddle's speed and acceleration at impact. The impact law, the period-1 orbit, the window of decelerating impacts, the mirror law and blind juggling are all established [6, 10, 12, 14]. The chapter writes the linearized impact map in closed form, with a determinant of e^2 whatever the paddle's waveform, checks the application's numerical Jacobian against it to 6.4×10^{-9}, and measures the same eigenvalues on the running engine with the arm in the loop. It also shows that a smooth periodic stroke always has a slowing branch, so the speeding-up side of the window cannot be held, and that near both edges the bounce-to-bounce spread of the restitution, not the eigenvalue, decides the outcome. The simulator is a 20-state model in which every impact is located inside its step. It drives the collection's six-axis arm module, which moved into the shared framework without changing a bit of its first user's runs, and a sensor boundary gives the laws the impact times and the paddle's own encoders but never the ball.

Section 15.2 traces juggling machines and the bouncing-ball map. Section 15.3 gives the ball, the impact, the arm and the sensors as simulated. Section 15.4 derives the impact map and its window of stable juggling and measures both with the arm in the loop. Section 15.5 gives the sinusoid, the once-per-bounce law and the mirror law as coded. Section 15.6 reports the nine cases and the challenge as measured, each case with the briefing and completion rule of Section 1.6.4. Section 15.7 describes the real impact sensors, arms and controllers, Section 15.8 states the limitations and Section 15.9 draws the conclusions.

15.2 Bouncing Balls, Return Maps and Juggling Machines

The physics came first as a problem in vibration. A ball bouncing on a table driven sinusoidally was analyzed in 1982 as a map from one impact to the next, under the simplification that the table's travel is small against the ball's flight, and the map was found to pass through a sequence of period doublings into chaotic motion as the driving amplitude grows [6]. The same system entered the teaching laboratory when a tabletop experiment reproduced that route to chaos in a physics teaching journal [7], and it serves as a worked example in the standard text on Poincaré maps and their bifurcations [8]. The mechanics of the impact itself, Newton's restitution and the treatment of instantaneous velocity jumps in rigid bodies, belongs to the theory of nonsmooth mechanical systems [2].

Robots took up juggling at the end of the 1980s as a laboratory for manipulation through intermittent contact. A one-joint robot that batted a puck sliding on an inclined plane was controlled by a family of mirror algorithms, in which the robot's motion is a distorted reflection of the puck's [14], and the same group carried the mirror law to planned juggling and catching [9]. In 1993 a vision-guided arm juggled two balls in space with one paddle [15], while a paddle moved on a fixed periodic trajectory was shown to juggle a ball open loop, with the stability coming from the paddle's motion at impact rather than from feedback on the ball [10]. A wider program of dynamic manipulation without grasping, in which a robot moves an object through the object's own dynamics, followed later in the decade [5].

The question of how little sensing a juggler needs was answered next. Sensorless stabilization of bounce juggling by a periodic paddle motion was analyzed and demonstrated [13], and a planar juggler that sensed only the instants of its impacts used them as feedback to strengthen the stability of its rhythm [11]. A robot that never measures the ball at all, a slightly concave paddle on a linear actuator moving on a fixed periodic trajectory, kept a ball bouncing in three dimensions in 2012 [12].

Table 15.1 Signals by Role. Ranges Are Those the Interface Offers

Symbol	Role	Default	Range
h	CV, apex of each flight	—	—
h^*	SP, target apex	0.50 m	0.20 to 0.70 m
x	CV of the tilt law, ball across the face	0	SP 0
z_p	MV, paddle height (Manual)	0	±0.15 m
β	MV, face tilt (Manual, tilt law)	0	±5°
$q_{c,i}$	MV, six joint commands	per law	joint ranges
$x, z, \dot{x}, \dot{z}$	states, each of two balls	per case	—
$q_i, \dot{q}_i$	states, six joints and rates	juggling pose	ranges
$t_k, p_k, \dot{p}_k$	measured, impact switch and encoders	each impact	—
$\hat{x}, \hat{z}$	measured, camera frame	30 Hz, 40 ms	10 to 120 Hz
e_k	hidden, restitution of impact k	$e_0 = 0.80$	—
Δe	DV, a different ball	0	−0.20 to +0.10
β_m	DV, face mounted crooked	0°	±3°
—	DV, camera dropout, 0.5 s	—	event
—	DV, flick across, 5 cm/s	—	event
n	output, consecutive bounces	0	—

HISTORY 1989: A Robot Bats a Puck Up a Slope

In May 1989 a robot that juggled by batting was presented at a robotics conference in Arizona [14]. Its puck slid on an inclined plane, and a bar with a billiard cushion, turning in that plane, struck the puck again and again. An oscillator inside the puck and a grid buried in the plane worked like a large digitizing tablet that gave the puck's position. An observer estimated the puck's velocity and passed it to the planning node at 1 kHz, where the mirror algorithm turned the puck's motion into the bar's reference, and the motor controller issued torque commands at 2 kHz. The controller was designed so that the juggling pattern was an attracting periodic orbit of robot and puck together [14]. The mirror law of Equation (15.13) reads a camera instead of a tablet.

Walking machines rest on the same mathematics. A passive biped walks down a slope on a stable gait with no actuation, its stability read from the step-to-step map [3], and the method of Poincaré sections, extended to systems with impulse effects, was used to prove asymptotically stable walking under feedback [4]. A general framework for hybrid systems with flows and jumps counts the bouncing ball among its examples [1]. The simulator gathers these pieces on one arm, so that the map, the drives and the sensors can be changed one at a time.

15.3 Ball, Paddle and Arm as Simulated

The simulated rig is the six-axis arm of the table-tennis robot app (https://apmonitor.com/apps/table-tennis-robot/), set on a pedestal and holding a paddle face up under a ball, as shown in Figure 15.1. The two applications are one robot with two jobs. The table-tennis robot strikes a ball to send it somewhere, and this one keeps a ball on one orbit. Volume 20, Chapter 11 describes the arm's kinematics and servos from the striking side. The frame has x across the paddle, along the arm's reach, and z up, both measured from the paddle's face at the middle of its stroke, which is the juggling plane. Heights a student reads, the apex and the target, are the ball's lowest point above that plane. The roles of the signals are listed in Table 15.1. Every constant of the ball, the camera and the arm and every threshold is a teaching value, and the application's model dialog and Instructions page say so.

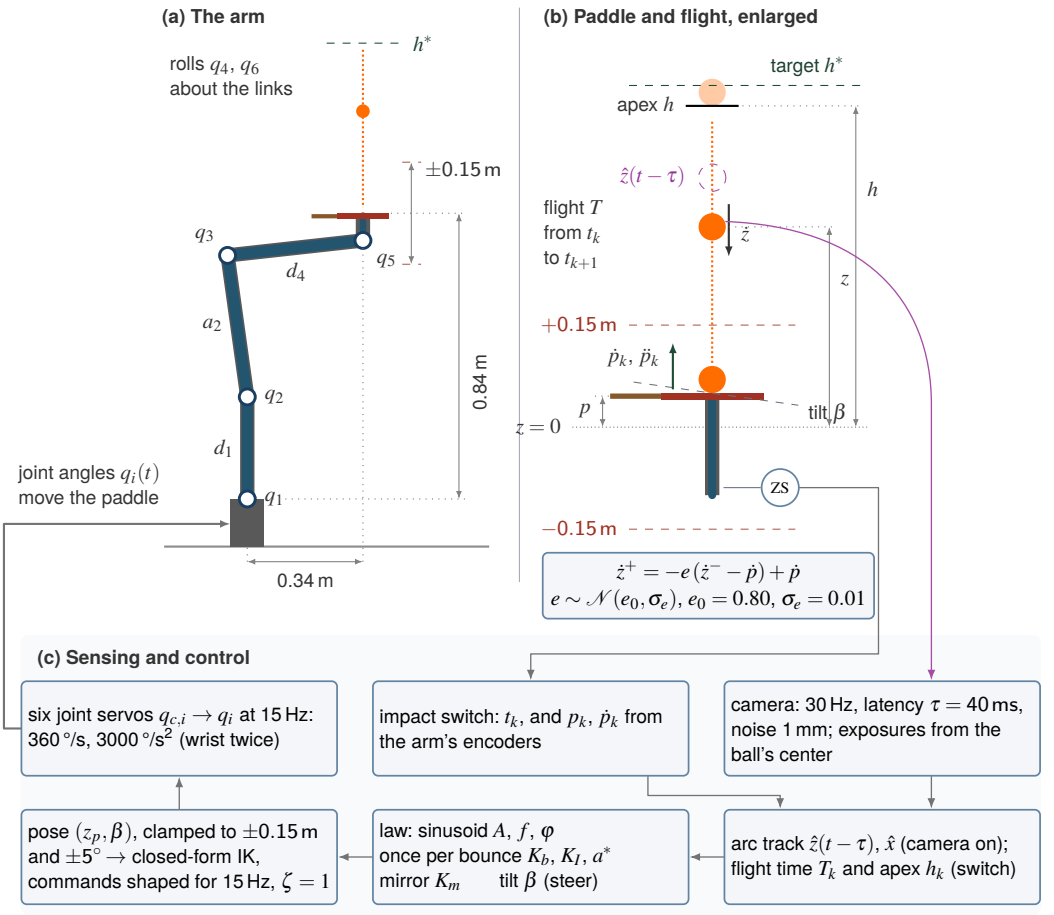


Figure 15.1 The juggling rig, panels (a) and (b) in the layout of the application's Rig view. (a) The six-axis arm on its pedestal at the juggling pose, drawn to scale from its forward kinematics, with the chain lengths d_1, a_2, d_4, the shoulder, elbow and wrist pitch joints q_2, q_3, q_5, the base turn q_1, the paddle's ± 0.15 m stroke and the ball's flight to the target apex h^*. (b) The paddle and the flight enlarged. The ball at height z and speed $\dot{z}$ meets the face at height p moving at $\dot{p}_k$ and accelerating at $\ddot{p}_k$, and the impact law turns $\dot{z}^-$ into $\dot{z}^+$ with a restitution drawn about e_0. Each flight of duration T rises to an apex h against the target h^*, the face can be tilted by β, and the camera sees the ball as $\hat{z}(t-\tau)$, a latency late. The impact switch ZS rings at each impact. (c) The camera and the switch feed the track, a law turns it into a pose, closed-form inverse kinematics and shaped commands turn the pose into joint commands, and six joint servos move the arm. Panels (b) and (c) are not to scale.

15.3.1 Flight

The ball is a solid rubber sphere of radius $r = 20$ mm and mass $m = 40$ g, and between impacts it flies under gravity and a quadratic drag, as shown in Equation (15.1).

$$\ddot{x} = -k_D\,|\mathbf{v}|\,\dot{x}, \qquad \ddot{z} = -g - k_D\,|\mathbf{v}|\,\dot{z}$$
$$k_D = \frac{\rho\, C_D\, \pi r^2}{2m} \tag{15.1}$$

where $\mathbf{v} = (\dot{x}, \dot{z})$, $g = 9.81\,\mathrm{m/s^2}$, $\rho = 1.2\,\mathrm{kg/m^3}$ and $C_D = 0.40$, which give $k_D = 0.0075\,\mathrm{m^{-1}}$. At the 3.13 m/s with which a 0.50 m bounce leaves the paddle, the drag is 0.75 % of gravity. The ball is

dense for a measured reason. A table-tennis ball of 2.7 g, with $k_D = 0.126\,\mathrm{m}^{-1}$, feels about an eighth of its weight in drag at that speed, loses 5.8 % of a 0.50 m rise and returns to the paddle with 88.8 % of the energy it left with. On a still paddle its first six apex ratios read e^2 between 0.56 and 5.7 % low, so the geometric series of case 2 would measure the air as much as the impact. The rubber ball reads e^2 between 0.04 and 0.39 % low.

15.3.2 The Impact, Located Inside Its Step

The face is a flat disc of radius 75 mm with unit normal $\mathbf{n}$, moving at the velocity $\mathbf{u}$ of the paddle's center. An impact keeps the ball's velocity along the face and reverses its normal velocity relative to the face with Newton's restitution [2], as shown in Equation (15.2).

$$\begin{aligned} v_n^{+} &= -e_k\,(v_n^{-} - u_n) + u_n, \qquad v_t^{+} = v_t^{-} \\ e_k &= \min\bigl(0.98,\ \max(0.05,\ e_0 + \Delta e + \sigma_e\,\xi_k)\bigr) \end{aligned} \tag{15.2}$$

where $v_n = \mathbf{v}\cdot\mathbf{n}$ and $u_n = \mathbf{u}\cdot\mathbf{n}$, ξ_k is a standard normal number drawn for each impact from a seeded stream, $e_0 = 0.80$ and $\sigma_e = 0.01$. With a level face the law is $\dot{z}^{+} = -e(\dot{z}^{-} - \dot{p}) + \dot{p}$ in the vertical, and a tilted face sends part of the impulse across, which is how a tilt steers the ball. The paddle's velocity is the arm's tool velocity from its Jacobian. The ball and the face are tested after every step. When the gap between the ball's surface and the face changes sign inside a step of length Δt, the fraction $f = s_0/(s_0 - s_1)$ of the step at which it closes follows from the gaps s_0 and s_1 at the two ends. Ball and face are both interpolated linearly to that fraction, the ball is set on the face, and after Equation (15.2) acts at that instant the ball's flight is integrated over the remaining $(1-f)\Delta t$ with Equation (15.1). A ball that approaches the face slower than 5 mm/s is taken to lie on it, and the juggle is then dropped. A ball whose center crosses the face's plane outside its rim has walked off, and once it falls more than 0.30 m below the juggling plane it is dropped. Either drop latches, stops the run and is reported in the status line.

Over every case as shipped and at half the step, every one of the impacts is located inside the step that holds it, the law holds to 2.7×10^{-15} m/s, and the ball's kinetic energy changes by the paddle's work less the restitution's loss to 10^{-14}. The steps themselves run 0.54 to 1.80 mm into the face before the locator acts, and no logged step leaves the ball in it. Under a paddle shaken at 20 Hz with an amplitude of 1 mm, the steps run 0.889 mm into the face and the smallest logged gap is +0.0001 mm.

15.3.3 The Arm That Holds the Paddle

The paddle is the tool of the collection's servo-level six-axis arm, `shared/src/framework/core/arm.js`. The table-tennis robot wrote the module, and it moved into the shared framework unchanged in arithmetic when this application became its second user. The table-tennis robot's ten scenarios logged bit-identical files before and after the move, shipped and with suggested settings. The juggler commands the arm through the same joint servos and the same command shaping, so the two applications run one arm. The kinematic chain is anthropomorphic and ends in a spherical wrist, as shown in Equation (15.3).

$$\begin{aligned} R &= R_z(q_1)\,R_y(q_2+q_3)\,R_z(q_4)\,R_y(q_5)\,R_z(q_6) \\ \mathbf{n} &= R\,\mathbf{e}_z \\ \mathbf{p}_t &= \mathbf{b} + d_1\mathbf{e}_z + a_2 R_z(q_1) R_y(q_2)\,\mathbf{e}_z \\ &\quad + d_4 R_z(q_1) R_y(q_2{+}q_3)\,\mathbf{e}_z + d_6 R\,\mathbf{e}_z \end{aligned} \tag{15.3}$$

where $\mathbf{p}_t$ is the paddle's center, $d_1 = 0.30\,\mathrm{m}$, $a_2 = 0.42\,\mathrm{m}$, $d_4 = 0.40\,\mathrm{m}$ and $d_6 = 0.08\,\mathrm{m}$, and the base $\mathbf{b}$ stands 0.34 m behind and 0.84 m below the middle of the stroke. That pose was chosen because a face-up paddle is bound by the wrist pitch's $\pm 125^\circ$, and from it every joint stays at least 14.3° inside its range over the ± 0.15 m stroke and $\pm 5^\circ$ of tilt, the wrist pitch at the bottom of the stroke

binding first. At the middle of the stroke the joints sit at $q_2 = -7.9°$, $q_3 = 91.6°$ and $q_5 = -83.7°$, the forearm across and the wrist pointing up. Each joint is a second-order drive with limits, as shown in Equation (15.4).

$$\begin{aligned} \ddot{q}_i &= \mathrm{sat}_{\ddot{q}_{max,i}}\left(\omega_j^2\,(q_{c,i}(t) - q_i) - 2\zeta_j\omega_j\,\dot{q}_i\right) \\ |\dot{q}_i| &\le \dot{q}_{max,i} \end{aligned} \tag{15.4}$$

where $\omega_j = 2\pi f_j$ with $f_j = 15\,\mathrm{Hz}$ and $\zeta_j = 1$ by default, and $q_{c,i}(t)$ is the latest command carried along its own rate between 5 ms updates. Speed and acceleration are capped at 360 °/s and 3000 °/s^2 for the first three axes and at double those values for the wrist. Approaching the end of its range, a drive brakes inside its acceleration limit so as to stop just short of it, and the range is a hard backstop in the plant. With the face level, the arm can accelerate the paddle vertically at 26, 19 and 13 m/s^2 at the bottom, middle and top of the stroke. No case as shipped reaches a range end or a reach limit, and outside the mirror law no drive reaches a speed or acceleration limit. On the stable sinusoid the busiest drive uses 29 % of its speed limit and 43 % of its acceleration limit.

15.3.4 What the Laws Can Know

The impact switch rings at every impact with its time t_k, and the arm's encoders give the paddle's height p_k and speed $\dot{p}_k$ at that instant. It does not say where across the face the ball struck. A blind juggler has nothing else, and two rings give a flight time $T_k = t_{k+1} - t_k$, from which the apex of the flight just ended follows by Equation (15.5).

$$h_k = p_k + \frac{v_k^2}{2g}, \qquad v_k = \frac{p_{k+1} - p_k}{T_k} + \frac{g\,T_k}{2} \tag{15.5}$$

The camera exposes each ball's center at exact multiples of $1/f_{cam}$, with $f_{cam} = 30\,\mathrm{Hz}$, interpolating the ball inside the step that holds the instant, adds 1 mm of Gaussian noise per axis, and delivers each frame $\tau = 40\,\mathrm{ms}$ after its exposure. During a dropout no exposure is delivered. The noise stream draws four numbers at every exposure, whether or not the camera is on, so that a run replays bit for bit. The laws read the camera through an arc track. The frames since the ball's last impact, at most 24, together with the paddle's height at that impact, are fitted by least squares with a ballistic arc in height (gravity known, drag left out) and a straight line across, and until a frame of the new flight arrives the arc leaves the impact at the speed the impact law predicts with $e^* = 0.80$ and the paddle's encoder speed. A law that acts on the camera reads this arc at $t - \tau$, the ball as the latest frame saw it carried along its flight, which the Rig view draws as a ghost ball behind the true one. No controller reads the simulator's truth, which the exported file carries in columns prefixed `truth_`.

15.3.5 Parameters and Numerics

Table 15.2 lists every parameter. The 20 states, each of two balls' position and velocity and the arm's six joint angles and rates, move forward in fixed fourth-order Runge-Kutta steps of 0.5 ms, in which a ball at 3 m/s moves 1.6 mm. The laws and the joint commands run every 5 ms, the camera exposes at its own instants and the historian logs at 100 Hz. Each step executes the laws, integrates, applies the rate clamp and range backstop, resolves impacts, records exposures and logs, in that order, which is the engine step of Section 1.6.1 with the impact resolved inside it. Halving the step to 0.25 ms moves the still paddle's apex ratios by 0.0002 % and the sinusoid's mean apex over 25 s by less than 0.0001 mm. Each case lets the ball go from rest at $t = 0$, its bottom at the target apex (35 cm in case 7) over the middle of the face, or 0.09 m to either side with two balls, the second let go half a bounce later. The arm starts on the law's reference at $t = 0$, or at the middle of the stroke in Manual, and the sinusoid's clock is set so that the stroke is at φ when the ball first arrives. The restitution and the camera noise come from seeded streams, so a run repeats exactly whether its case is opened first, chosen after another case, or restarted with Reset.

Table 15.2 Model Parameters. A Is Adjustable in the Model Dialog, V Stated in the Model Dialog but Fixed, H Hidden in the Code

Symbol	Meaning	Value	
Ball and impact			
e_0, σ_e	restitution, mean and spread	0.80, 0.01	A
k_D	drag factor (C_D 0.40, ρ 1.2 kg/m^3)	0.0075 m^{-1}	A
r, m	ball radius, mass	20 mm, 40 g	V
r_p	paddle face radius	75 mm	V
—	approach at which the ball rests	5 mm/s	V
—	drop height below the plane	0.30 m	H
Arm (the table-tennis robot's)			
$\dot{q}_{max}$	speed limit, axes 1 to 3 (wrist ×2)	360 °/s	A
$\ddot{q}_{max}$	acceleration limit, axes 1 to 3 (wrist ×2)	3000 °/s^2	A
d_1, a_2, d_4, d_6	chain lengths	0.30, 0.42, 0.40, 0.08 m	H
—	ranges, axes 1 to 6 (±°)	170, 100, 150, 200, 125, 200	H
b	base, behind and below the stroke	0.34, 0.84 m	V
—	stroke and tilt limits	±0.15 m, ±5°	V
f_0, ζ_0	drive tuning the commands assume	15 Hz, 1	V
Sensors			
f_{cam}, τ	camera rate, latency	30 Hz, 40 ms	A
σ_{cam}	camera noise per axis	1 mm	A
—	frames in the arc track	24	H
—	seed (restitution, camera)	1	A
Numerics			
Δt, T_c	integration step, control period	0.5, 5 ms	H
—	historian	100 Hz	H

15.3.6 Assumptions

A rigid ball in one plane. The ball does not spin and stays in the arm's vertical plane. A real ball spins after a tilted or off-center hit and drifts out of the plane, and its spin changes the next bounce.

Instantaneous Newton impacts. The restitution does not depend on the impact speed or on where the ball lands, and the face does not grip the ball. Real restitution falls as impacts get faster, and a rubber face grips and spins a ball, so the bounce-to-bounce spread stands in for all of this [2].

A flat face. A drifting ball is steered back only by tilting the face on the camera's estimate. A slightly concave face, as on the blind juggler of [12], would steer it back with no sensor.

A servo-level arm. Positions and orientations follow exact kinematics, while inertia, torque limits and flexibility are left out, and the arm's capacity does not change with what it carries.

Ideal sensors apart from rate, latency and noise. The switch reports every impact exactly, and the camera has no blur, occlusion or lost frames outside a dropout.

15.4 One Bounce at a Time

15.4.1 The Period-1 Orbit

A bounce that rises h above the point of impact leaves at $v = \sqrt{2gh}$ and lasts $T = 2v/g$. To send it back up at the same speed, the paddle must meet it moving up at the speed that replaces the restitution's loss, as shown in Equation (15.6).

$$v = -e\,(-v - u^*) + u^* \quad \Longrightarrow \quad u^* = v\,\frac{1-e}{1+e} \tag{15.6}$$

For $h = 0.50\,\text{m}$ and $e = 0.80$ this is $v = 3.13\,\text{m/s}$, $T = 0.639\,\text{s}$ (1.57 Hz) and $u^* = 0.348\,\text{m/s}$. The flight time fixes the frequency of any stroke that meets every bounce, and the frequency fixes the apex. At 1.655 Hz, the stroke of most cases, a bounce lasts 0.604 s, rises 0.448 m above its impact and needs the paddle at 0.33 m/s, or 0.48 m/s for a ball whose restitution has fallen to 0.72.

15.4.2 The Map and Its Linearization

Let the paddle repeat with period T, and let τ_k be the offset of impact k from its nominal instant and v_k the ball's speed after it. The next impact is where the ball's parabola meets the paddle again, and the impact law there gives the next speed. This is a map from one bounce to the next, and the period-1 orbit is its fixed point, a cyclic steady state in the sense of Section 1.4.9. Linearizing about the period-1 orbit, with the paddle meeting the ball at u^* and accelerating at $a = \ddot{p}(t_k)$, gives Equation (15.7).

$$\begin{bmatrix} \delta\tau_{k+1} \\ \delta v_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \dfrac{1+e}{g} \\ (1+e)\,a & e^2 + \dfrac{(1+e)^2 a}{g} \end{bmatrix}}_{J} \begin{bmatrix} \delta\tau_k \\ \delta v_k \end{bmatrix} \tag{15.7}$$

The first row says that a faster launch comes back later, by $T/(u^* + v) = (1+e)/g$ per unit of speed. The second says that a late ball meets a paddle whose speed has changed by $a\,\delta\tau$, and that a ball launched faster by δv also arrives faster by $e\,\delta v$ and leaves the next impact faster by $e^2\delta v$ when the paddle does not accelerate. Only the paddle's speed and acceleration at the instant of impact enter, so J is the same for a sinusoid or any other periodic waveform, at any frequency. Its determinant and trace are Equation (15.8).

$$\det J = e^2, \qquad \operatorname{tr} J = 1 + e^2 + \frac{(1+e)^2}{g}\,a \tag{15.8}$$

The eigenvalues solve $\lambda^2 - \operatorname{tr} J\,\lambda + e^2 = 0$. One reaches $+1$ where $\operatorname{tr} J = 1 + e^2$, at $a = 0$, and one reaches -1 where $\operatorname{tr} J = -(1+e^2)$, so the orbit is stable for paddle accelerations at impact in the window of Equation (15.9) [6, 10].

$$-\frac{2g\,(1+e^2)}{(1+e)^2} < \ddot{p}(t_k) < 0 \tag{15.9}$$

At $e = 0.80$ the window runs from $-9.9311\ \mathrm{m/s^2}$ to 0, and at $e = 0.72$ its lower edge moves to $-10.070\ \mathrm{m/s^2}$. Where $|\operatorname{tr} J| < 2e$, from $a = -g$ to $a = -g(1-e)^2/(1+e)^2$, or $-9.81\ \mathrm{m/s^2}$ to $-0.121\ \mathrm{m/s^2}$, the eigenvalues are complex with modulus exactly e, so an error shrinks by a fifth every bounce whatever the stroke. Near the two edges they are real and one of them grows toward 1, to 0.970 at $-9.9\ \mathrm{m/s^2}$ and 0.949 at $-0.05\ \mathrm{m/s^2}$. Outside the window it is 1.977 at $+2\ \mathrm{m/s^2}$.

> **+ DID YOU KNOW? Never With a Rising Arm**
>
> On a planar juggler that sensed only the times of its impacts, sustained stabilization in the laboratory needed the arms' acceleration at impact tuned negative, and it could never be achieved with a positive one [11]. The window of Equation (15.9) ends at the same zero.

The application computes the same Jacobian numerically, by central differences on its own impact map of an ideal sinusoidal paddle, `paddle-juggler/src/poincare.js`, which finds the period-1 orbit by Newton's method and draws the window on the Map view. Over 26 accelerations from -14 to $+4\ \mathrm{m/s^2}$, at 1.566 and 1.655 Hz, its largest eigenvalue modulus agrees with Equation (15.8) to 6.4×10^{-9}, its determinant reads 0.6400 at every one, and its bisected edges are $-9.9311\ \mathrm{m/s^2}$ and 0, as shown in Figure 15.2a. On a finer grid of 386 accelerations the largest difference, 1.7×10^{-5}, falls at $-9.81\ \mathrm{m/s^2}$, where the two eigenvalues meet on the real axis.

15.4.3 The Window With the Arm in the Loop

The ideal map assumes a paddle that follows its waveform exactly. The engine's paddle is the arm's tool, driven through inverse kinematics and six servos, so the eigenvalues were measured on the

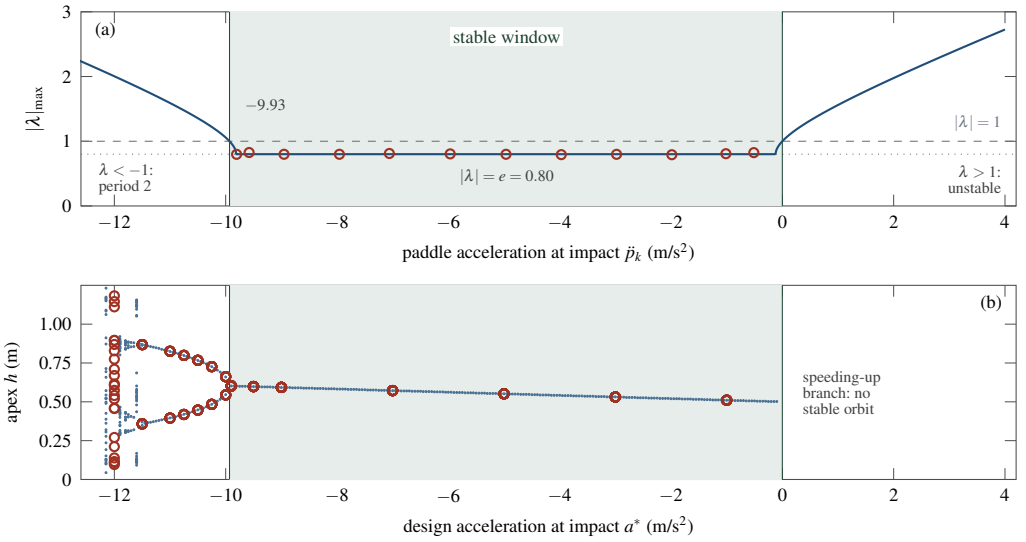


Figure 15.2 The window of stable juggling at $e = 0.80$. (a) The largest eigenvalue modulus of the impact map against the paddle's acceleration at impact, from the application's numerical Jacobian (line), with the window between $-9.93\ \mathrm{m/s^2}$ and 0 shaded, and the eigenvalues measured on the running engine with the arm in the loop (circles), e exact and no drag, from the apex errors after a 2 cm/s kick. (b) The apexes of the last 40 of 300 bounces of the ideal map (dots) and of the engine's flights over the last 15 s of a 40 s run (circles), for strokes at 1.566 Hz designed for each acceleration. Past the lower edge the orbit splits into two apexes, then four, then scatters.

engine as well. For each design acceleration from -0.5 to $-9.7\ \mathrm{m/s^2}$, a sinusoid at 1.566 Hz was run for 40 s with the restitution exact and no drag, the ball was given a 2 cm/s kick after an apex, and the next 15 apex errors were fitted with the second-order recursion $\delta h_{k+1} = c_1\,\delta h_k + c_2\,\delta h_{k-1} + c_0$, whose coefficients are $\mathrm{tr}\,J$ and $-\det J$ for any scalar output of a two-state linear map. The measured eigenvalue moduli lie between 0.79 and 0.83 against the closed form's 0.80, the determinants between 0.63 and 0.68 against 0.64, and the traces within 0.045 of Equation (15.8) at the measured acceleration, as shown in Figure 15.2a. Beyond the edge the arm's orbit splits as the ideal map's does, into apexes of 0.544 and 0.662 m at a design of $-10\ \mathrm{m/s^2}$ against 0.547 and 0.660 m, 0.446 and 0.767 m at -10.5 against 0.447 and 0.766, and 0.395 and 0.825 m at -11 against 0.396 and 0.825, as shown in Figure 15.2b. The ideal map doubles again near $-11.5\ \mathrm{m/s^2}$ and scatters beyond, and at $-12\ \mathrm{m/s^2}$ the engine's apexes range from 0.096 to 1.182 m. The edges hold with the servos in the loop, and at half the step the arm's period-1 and period-2 orbits are unchanged, their accelerations at impact within $0.01\ \mathrm{m/s^2}$.

15.4.4 Why a Slowing Paddle Is Stable, and Why It Must Be

A ball that comes back late meets a paddle that has already slowed. It gets less of a push, leaves lower and comes back sooner, and a ball that comes back early meets a faster paddle and is sent higher and later. Every bounce is pulled back onto the beat, which is the negative feedback hidden in Equation (15.7), and it needs no sensor. Slowing too hard overcorrects, so a late ball comes back early by more than it was late, and the orbit alternates between a high and a low bounce once an eigenvalue passes -1. A riderless bicycle is self-stable in the same way only inside a band of its forward speed (Volume 7, Chapter 9).

Pause. Try Exercise 15.1 before reading on.

Table 15.3 Controller Settings, Defaults and Ranges. Settings Marked adv. Are Shown in the Advanced View

Symbol	Setting	Default	Range
—	law: sinusoid, once per bounce, mirror law or two balls	sinusoid	four
h^*	target apex	0.50 m	0.20–0.70
A, f	stroke amplitude, frequency	6.3 cm, 1.655 Hz	0–15, 1–2.5
φ	stroke phase at first arrival	55°	±180
K_b	bounce-to-bounce gain, (m/s)/m	0.8	0–3
K_I	integral per bounce, (m/s)/m	0.2	0–1.5
a^*	design acceleration at impact	-5 m/s^2	-14 to 4
K_m	mirror gain	1.5	0–2.5
—	camera, tilt law	on, level	off, steer
f_j, ζ_j	joint bandwidth, damping (adv.)	15 Hz, 1	2–30, 0.2–2
z_p, β	manual height and tilt	0, 0°	±15 cm, ±5°
e^*	restitution the laws assume	0.80	fixed
—	mirror target filter	12 Hz, ζ 1	fixed
—	largest commanded impact speed	1.5 m/s	fixed
—	amplitude slew	0.10 m/s	fixed
—	fade after a mode or law change	0.5 s	fixed
—	manual follower	1 m/s, 10 m/s^2	fixed

For a sinusoid $z_p = A\sin\omega t$, the paddle has the speed u^* at two phases of each cycle, one speeding up and one slowing down, with the acceleration at impact $\pm\omega\sqrt{A^2\omega^2 - u^{*2}}$. At 1.655 Hz the window's edges correspond to strokes of 3.17 and 9.71 cm. The speeding-up branch is unstable. A smooth periodic stroke that reaches u^* on its way up reaches it again while slowing, so a ball knocked off the speeding-up orbit settles on the slowing one. With the arm in the loop, the restitution exact and no drag, a stroke designed for +2 m/s^2 at 1.566 Hz is the 4.10 cm stroke also designed for -2 m/s^2, and the ball settles on the slowing branch at -2.01 m/s^2. The planning script that sized this application iterated a paddle on a fresh parabola about each nominal impact, which has no second branch, and its rows (converging from -2 to -8 m/s^2, wandering at 0, lost at $+2$ and at -10 and below) are reproduced by the application's test of that map.

Pause. Try Exercise 15.2 before reading on.

15.5 The Laws as Coded

Every 5 ms the active law gives the paddle's pose as a function of time near the present, $P(s) = (x_p, z_p, \beta)$, its place across (two balls only), its height and its tilt. The pose is clamped to ±0.15 m in height, ±0.12 m across and ±5° of tilt, turned into joint angles by the arm's closed-form inverse kinematics for the paddle's center and normal (the roll about the normal held at zero), and differentiated by five-point central differences with a 2 ms spacing to give each joint's reference r and its first three derivatives. The command sent to each drive is the servo inverse of Equation (15.10).

$$q_c = r + \frac{2\zeta_0}{\omega_0}\dot{r} + \frac{\ddot{r}}{\omega_0^2}, \qquad \omega_0 = 2\pi\cdot 15\,\text{Hz}, \quad \zeta_0 = 1 \tag{15.10}$$

together with its rate, so that a drive tuned as commissioned follows the law without lag while it stays inside its limits, and one retuned away from 15 Hz falls behind it. Measured with a 1 mm stroke, the shaped tool follows within 0.6 % in amplitude and with less than 0.1° of phase lag up to 6 Hz, while the unshaped servos alone reach their -3 dB point at 9.9 Hz, and drives detuned to 6 Hz under the 15 Hz shaping at 4.4 Hz. This is the table-tennis robot's pipeline. None of the laws uses the framework's velocity-form PID of Section 1.6.2, and the settings are listed in Table 15.3.

15.5.1 The Sinusoid, With No Sensor

The open-loop stroke is $z_p = A\sin\theta(t)$ with $\dot{\theta} = 2\pi f$, its amplitude following the slider at 0.10 m/s, and its phase set so that the stroke is at φ when the ball, let go from rest at the target apex, first arrives

(0° is the middle of the up-stroke and 90° its top). It reads no sensor. On its slowing branch it meets the ball at the design acceleration of Equation (15.11), which the scorecard and the Map view report.

$$a^*_{sine} = -\omega\sqrt{A^2\omega^2 - u^{*2}}, \qquad u^* = \frac{g}{2f}\,\frac{1-e}{1+e} \tag{15.11}$$

For two balls the paddle moves across as $x_p = -0.09\cos(\theta - \varphi)$ and up as $z_p = A\sin(2\theta - \varphi)$, so that it strokes at $2f$ and each ball, 0.18 m from the other, meets it every other stroke. Each ball's bounce then lasts $1/f$, and the design acceleration uses u^* for that bounce with the stroke's $\omega = 4\pi f$.

15.5.2 Once per Bounce, on the Impact Switch

The once-per-bounce law is a discrete-time loop whose sample is an impact, in the spirit of feedback from impact times alone [11]. At each ring of the switch it forms the apex error $e_{h,k} = h^* - h_k$ from Equation (15.5) and sets the paddle's speed for the next impact, as shown in Equation (15.12).

$$\begin{aligned} I_k &= I_{k-1} + K_I\, e_{h,k} \\ \hat{u}_{k+1} &= u^*(h^*) + K_b\, e_{h,k} + I_k \\ u_{k+1} &= \min\bigl(1.5\,\mathrm{m/s},\ \max(0,\ \hat{u}_{k+1})\bigr) \end{aligned} \tag{15.12}$$

where $u^*(h^*)$ is Equation (15.6) for the target with $e^* = 0.80$. The integral is not limited, and only the speed it sets is clamped. The next arrival is predicted from the measured incoming speed with e^*, and a quintic in height carries the paddle from its present position, speed and acceleration to the juggling plane at that instant, arriving at u_{k+1} and decelerating at a^*, and continues on that parabola if the ball is late. The law needs no camera. The restitution of impact k that it reports, $\hat{e}_k = (v_k - \dot{p}_k)/(w_k + \dot{p}_k)$, from the launch speed v_k of the flight after it, the arrival speed $w_k = gT_{k-1} - v_{k-1}$ of the flight before it and the paddle's speed, is the e measured badge of the scene.

15.5.3 The Mirror Law, on the Camera

The mirror law moves the paddle as a scaled reflection of the ball the camera shows [9, 14], with a gain that rises when the seen apex $\hat{h}$ is low, as shown in Equation (15.13).

$$z_p = -\kappa\,\hat{z}(t-\tau), \qquad \kappa = \frac{1-e^*}{1+e^*} - K_m\,\frac{\hat{h}-h^*}{2\,(1+e^*)\,h^*} \tag{15.13}$$

with $\kappa \geq 0$ and $\hat{z}$ held at zero or above. For a camera that sees the ball at once and a paddle that follows the mirror target at once, the bounce-to-bounce eigenvalue of the apex is $1 - K_m$, so a gain of 1 corrects an error in one bounce and 2 is the edge. The target passes through a critically damped 12 Hz filter before the shaping of Equation (15.10), because its jumps at each impact would otherwise hold the drives on their limits. With the camera on, the tilt law turns the face to steer a drifting ball back, $\beta = -(\hat{x}/2T^* + \dot{\hat{x}})/\bigl((1+e^*)(v^*+u^*)\bigr)$, from the track's position and speed across and the target orbit's flight time, launch speed and impact speed.

15.5.4 By Hand, Transfers and the Status Line

In Manual the paddle's height follows its station, a slider, the arrow keys or a tap above or below the paddle, each press 2 cm, at up to 1 m/s and 10 m/s^2, with its joint commands carried along that motion, so a press uses at most 85 % of a drive's acceleration limit and the face stays level to better than 0.0001°. A switch to Manual leaves the stations where the law had the paddle, and a switch to Auto or a change of law runs the new law at once while the offset from the held paddle decays over 0.5 s on a minimum-jerk profile. The status line names DROPPED, WALKED OFF, REACH LIMIT, JOINT LIMIT, CAMERA STALE, TWO BALLS and BLIND, and none of them acts on the arm.

Table 15.4 The Nine Cases and the Challenge

#	Configuration and events	Teaching point
1	Manual, ball let go at 50 cm	reaction is too late
2	still paddle, then $\Delta e = -0.10$	restitution from e^2
3	sinusoid 6.3 cm, 1.655 Hz, first arrival at −135°	the first meeting
4	the same at 55°, amplitude swept on the Map view	the window
5	camera off, e 0.80 to 0.72 at 20 s	juggling blind
6	mirror law, K_m 0.5, tilt steering	latency costs height
7	once per bounce, $K_I = 0$, from 35 cm, e to 0.72 at 12 s	an integral per bounce
8	two balls, 2.2 cm at 3.2 Hz, 18 cm apart	one arm, two orbits
9	mirror law, $h^* = 0.20$ m, camera 120 ms late	honestly infeasible
C	60 s, e to 0.72 at 20 s, dropout at 40 s	everything at once

Table 15.5 Outcomes Read From the Running Engine

Case and condition	Result
1, untouched / press 30–90 ms early	dies after 29 impacts at 2.98 s / next apex 41.6–52.2 cm
2, e 0.80 / 0.70	ratios 0.650 / 0.499, their roots 0.806 / 0.707
3, −135° / 55°	dies at 2.28 s / 49 bounces in 30 s, 0.502 m ± 1.6 cm
4, 3.4–9 cm / 9.75 / 10 cm	hold at 5 seeds / scatters with no spread / 0.11–0.96 m
5, before / after 20 s	0.500–0.503 m / 0.489–0.491 m, none dropped of 6
6, K_m 0.5 / 1 / 1.5 / 2	0.374 / 0.442 / 0.462 / 0.472 m
7, P / PI / none, after 12 s	0.457 / 0.503 / 0.383 m
8, 2.2 cm / 1.5 cm	64 and 63 bounces in 40 s / walks off at 2.76 s
9, K_m 0.25 to 2.5	every gain dies, 1.39–2.38 s
C, shipped / suggested	dropped at 23.2 s, 35 / 99 bounces, 1.6 cm rms

15.5.5 Three Models

The simulation model is everything in Section 15.3. The analysis model is the impact map of an ideal paddle and a drag-free ball, whose linearization is Equation (15.7). The laws' own model is smaller still. They assume $e^* = 0.80$ and a drag-free arc, read the ball only through the switch and the camera, and are never told the true restitution, a change of ball or a crooked face.

15.6 Ten Cases, Measured

The cases are listed in Table 15.4 and their outcomes in Table 15.5. Every number below was measured with the engine stepped headless exactly as the page steps it, at the default seed unless a range of seeds is named, and a setting that a sentence changes was set before pressing Run. The apex is the ball's lowest point at the top of a flight. A bounce counts toward the score when its flight rises at least half the target, 25 cm at the 50 cm target, so the millimeter hops of a dying ball are impacts but not juggling. The application passes its 39 verification checks.

15.6.1 By Hand, and a Still Paddle

From 50 cm the ball meets the paddle at 3.13 m/s, 0.319 s after it is let go. Left alone, the paddle lets the bounces die, 33, 22, 14 and 9 cm, and only the first flight counts. A press of the up key moves the paddle 2 cm, and it sends the next bounce back to 41.6 to 52.2 cm only when it comes 30 to 90 ms before the impact. Pressed 15 ms before, the next apex is 35.9 cm, and pressed at the impact it is the untouched 33.2 cm. A simulated student who presses as the ball is seen to land, with a timing spread of 50 ms, keeps 2.6 counted bounces on average and loses all 40 trials within 2.9 to 8.0 s, while one who presses a steady 50 ms ahead with a spread of 25 ms keeps 43.8 on average and loses none of 40 in 30 s. The window rewards anticipation, which reaction cannot supply.

The still paddle of case 2 measures the ball. Its apexes fall 33.2, 22.1, 14.3, 9.2, 5.9 and 3.8 cm, each 0.64 to 0.67 of the one before with the bounce-to-bounce spread, 0.650 on average, and $\sqrt{0.650} = 0.806$ against the ball's 0.80. With the spread removed the ratios are 0.6375 to 0.6397

against $e^2 = 0.6400$, the small drag taking the rest. The ball lies still after 29 impacts in 2.98 s, a geometric series that fits infinitely many impacts into a finite time if nothing stops it, the Zeno solution of Section 1.4.7. A ball of restitution 0.70 keeps 0.499 of each rise and dies after 19 impacts in 1.88 s. The e measured badge, computed as the blind juggler would from two flight times and the paddle's speed, reads 0.80 to 0.82 over the first ten impacts.

15.6.2 The Sinusoid and the Window

The stroke of case 3 is fixed, and only the phase at which the ball first arrives changes. At −135°, with the paddle on its way down, the bounces die after 28 impacts in 2.28 s. At 55° the ball stays up, 49 bounces in 30 s with the apex at 0.502 m and a spread of 1.6 cm, each impact decelerating at 5.9 m/s^2 and the paddle 5.4 cm above the juggling plane. Every first meeting from −80° to 150° in 5° steps ends on that orbit, even −60°, whose first impact comes while the paddle is still speeding up at 5.0 m/s^2. Only first meetings from 155° round to −85°, with the paddle going down, let the bounces die, in 2.1 to 4.6 s. The orbit is an attractor with a large basin.

> **TRY IT IN THE APP** **3 · The Sinusoid**
>
> Load the scenario and press Run. The ball first meets a paddle on its way down, and the bounces die after 28 impacts in 2.28 s. Press Reset, set Phase at the first arrival to 55° and run again. The same stroke now keeps the ball up, 49 bounces in 30 s with the apex at 0.502 m.

Case 4 sweeps the amplitude at 1.655 Hz on the Map view. From 3.4 cm, slowing at 1.1 m/s^2 at impact, to 9 cm, slowing at 9.0 m/s^2, the orbit holds at all five seeds tried, with apex spreads of 1.1 to 4.9 cm caused by the ball's own spread. Past the lower edge, 9.75 cm (−9.97 m/s^2) scatters even with the spread removed, and 10 cm sends the apexes from 0.11 to 0.96 m. Near both edges the restitution's spread decides before the eigenvalue does. At 9.5 cm the paddle slows at 9.7 m/s^2, still inside the window, and the orbit holds with no spread (an apex spread of 0.7 cm) but scatters at four seeds of five. At 3.3 cm the stroke's top speed, 0.343 m/s, is only 3 % above the 0.334 m/s that a 1.655 Hz bounce needs with the ball's drag, and the orbit holds with no spread but dies at four seeds of five, in 6.7 to 14.6 s. At 3.2 cm, whose top speed is 0.333 m/s, and below, the stroke never reaches that speed and the ball dies with or without the spread. Raised to 11 cm while running, the apexes scatter from 0.18 to 0.98 m within 14 s, and lowered to 3 cm the ball lies on the paddle 3.7 s later.

15.6.3 Blind, Through a Change of Ball

Case 5 switches the camera off, so the only sensor left is the impact switch, and the sinusoid does not read that either. At 20 s the ball is changed for one of restitution 0.72, which needs the paddle at 0.48 m/s instead of 0.33. The orbit moves to a new phase by itself. Over seeds 1 to 6 the paddle's acceleration at impact moves from −5.75 to −5.87 before the change to −4.41 to −4.69 m/s^2 after it, the mean apex from 0.500 to 0.503 to 0.489 to 0.491 m, and the ball is never dropped. The deader ball comes off lower, returns a little early and meets a faster part of the stroke, and the frequency, which fixes the flight time, keeps the apex about 1 cm below where it was. The state of the default seed 9.95 s after the change is shown in Figure 15.3.



Figure 15.3 Juggling blind through a change of ball, case 5 at $t = 29.95$ s on the Rig view. The picture is a composite: the Rig view is cropped from the screen, with the header, the control panel and the trends cut, and the four tiles of the scorecard's This app row, from the Learn panel, are set beside it. The camera is off and the status line reads BLIND, the impact switch only. (1) The arm holds the paddle. (2) The last impact, at 29.906 s, still shows as a burst at the contact and an arrow for the ball's change of speed, 5.9 m/s. (3) The badges read 49 bounces, an apex of 49.1 cm, the paddle slowing at 4.8 m/s^2 and rising at 0.48 m/s at impact, a flight of 605 ms, and a restitution of 0.72 measured from the flight times alone for the impact at 29.30 s, against the 0.726 drawn for it. (4) Under 49 consecutive bounces, the count panel's 16 apex bars are all flights after the change at 20 s, the first 6.0 cm and the fifth 4.2 cm below the target outside the 3 cm band, the rest inside it. (5) The scorecard's tiles give an apex error of 1.8 cm rms over 46 flights, the present stroke's design acceleration of −4.6 m/s^2 for the deader ball, and a smallest gap at the log of 0.008 mm.

> **TRY IT IN THE APP** **5 · Blind**
>
> Load the scenario, whose camera is off, and press Run. Watch the apex strip and the badges across the change of ball at 20 s. Nothing measures the ball, yet the deader ball meets a faster part of the stroke, its apex settles about 1 cm lower, and it is never dropped.

15.6.4 The Mirror Law and Its Latency

Case 6 closes the loop through the camera, 40 ms late. The apex settles short of the target by an amount the gain shrinks, 0.374, 0.442, 0.462 and 0.472 m at gains of 0.5, 1, 1.5 and 2, with spreads of 1.0 to 1.3 cm, tighter than the sinusoid's 1.6 cm. With no latency the gains of 0.5 and 1.5 reach 0.455 and 0.485 m, so the latency costs energy, and at 80 ms the gain of 0.5 loses the ball in 11.9 s. Latency does not bring the bounce loop's oscillation to a lower gain. Measured as the lag-one correlation of the apex errors over six seeds, the bounce-to-bounce eigenvalue is -0.50 at a gain of 2 and -0.89 at 2.5 through the 40 ms camera and the target filter, and -0.73 and -0.99 with no latency, against the ideal -1 and -1.5, so at both latencies it nears -1 only at 2.5, where the spread grows to 2.1 cm. The mirror law strikes while the paddle is still speeding up, at 1.0 to 1.6 m/s^2 on average over the gains, outside the open-loop window, where only the feedback holds the orbit. Reversing with the ball at each bounce, it puts a drive on its acceleration limit for 8.3, 16.4, 18.0 and 18.6 ms a bounce at the four gains. Flicked across at 5 cm/s, the ball is steered back by the tilt law within 3.0 cm of the middle, and with the face held level it walks off the edge 2.0 s later.

15.6.5 An Integral That Runs Once per Bounce

Case 7 starts the ball at 35 cm on the once-per-bounce law with the camera off. The proportional part brings the apex to the target in four bounces, 0.390, 0.406, 0.449 and 0.502 m, and the PI runs exactly once per ring of the switch, 48 times for 48 rings. After the change of ball at 12 s the nominal speed u^* is too slow, and the proportional part settles 4 cm short, at 0.457 m, because it needs the error to act. With an integral of 0.2 per bounce the apex returns to 0.503 m, at the cost of an overshoot to 0.569 m on the climb from 35 cm, and with no gain at all it falls to 0.383 m. After the change the deader ball arrives early on the planned quintic and meets the paddle at about $-0.06\ \mathrm{m/s^2}$, near the window's upper edge, so the feedback rather than the stroke's own deceleration holds the orbit.

15.6.6 Two Balls, and a Camera Too Late

Two balls 18 cm apart share one arm in case 8, the paddle crossing between them with up to $9.5\ \mathrm{m/s^2}$ across at its turns and stroking 2.2 cm at 3.2 Hz. In 40 s they bounce 64 and 63 times at mean apexes of 0.492 and 0.493 m, spreads of 2.1 and 2.0 cm, every impact slowing, at 5.41 and $5.45\ \mathrm{m/s^2}$ on average against the $5.67\ \mathrm{m/s^2}$ the stroke was designed for. At 1.5 cm the balls fall off the beat and one walks off the edge after 8 impacts, 2.76 s into the run.

Case 9 is infeasible by design. A 0.20 m bounce lasts 0.404 s, and a camera 120 ms late shows each frame after 30 % of a bounce has passed. Every mirror gain from 0.25 to 2.5 in steps of 0.25 drops the ball, in 1.39 to 2.38 s, each time by the bounces shrinking until the ball lies on the paddle. The paddle rises for a ball that left its apex 120 ms earlier, meets it late and slow, and gives back less than the impact took. With the 40 ms camera the same law holds a 0.172 m apex, at 80 ms it lasts 9.10 s and at 100 ms 2.94 s. Without the camera a 3 cm sinusoid at 2.5 Hz keeps the ball up indefinitely at 0.223 m, decelerating at $6.5\ \mathrm{m/s^2}$ at impact. Information that arrives after it was needed cannot be bought back with gain. A robot that picks parts off a conveyor meets the same latency and moves each late camera report on by the belt travel since its capture (Volume 20, Chapter 10).

Pause. Try Exercise 15.3 before reading on.

15.6.7 A Minute of Juggling

The challenge juggles for 60 s at a 0.50 m target through a change of ball from 0.80 to 0.72 at 20 s and a camera dropout of 0.5 s at 40 s. The target is no drop and an apex within 3 cm rms of the target over every flight from 2 s, with the model and the disturbances as given, and the score is the count of consecutive bounces. Changing a model parameter other than the seed, a disturbance or the target during the challenge forfeits that attempt, and the run pauses at 60 s with its verdict. The configurations are compared in Table 15.6. As shipped, a 4.1 cm sinusoid at 1.60 Hz slows at only $2.1\ \mathrm{m/s^2}$ at impact. The deader ball needs the paddle at 0.50 m/s at that frequency, the stroke never moves faster than 0.41 m/s, and the ball is dropped about 3 s after the change at every seed from 1 to 10. The suggested stroke, 6.3 cm at 1.655 Hz, slows at $5.9\ \mathrm{m/s^2}$, has the margin for the deader ball, never reads the camera and passes at all ten seeds. The once-per-bounce law passes too, and the comparison shows what feedback adds here. Its integral holds the mean apex after the change at 0.497 to 0.500 m, where the blind stroke settles at 0.489 to 0.490 m, but its strokes then meet the deader ball at -0.05 to $+0.24\ \mathrm{m/s^2}$, with no deceleration to damp an error, and its spread grows to 1.5 to 2.4 cm against 1.2 to 1.6 cm. The bias that feedback removes is smaller than the spread it lets in. The suggested minute plays in 30.0 s of wall time at the top speed of 2× in a browser. Figure 15.4 draws the five configurations against the 3 cm target.

> **DID YOU KNOW? Feedback That Scores Worse**
>
> From the simulator: over the challenge minute at seed 1, the blind 6.3 cm sinusoid holds the apex to 1.6 cm rms and the once-per-bounce law to 2.5 cm, while the mirror law on the camera keeps 98 bounces and still fails at 7.6 cm (Table 15.6).

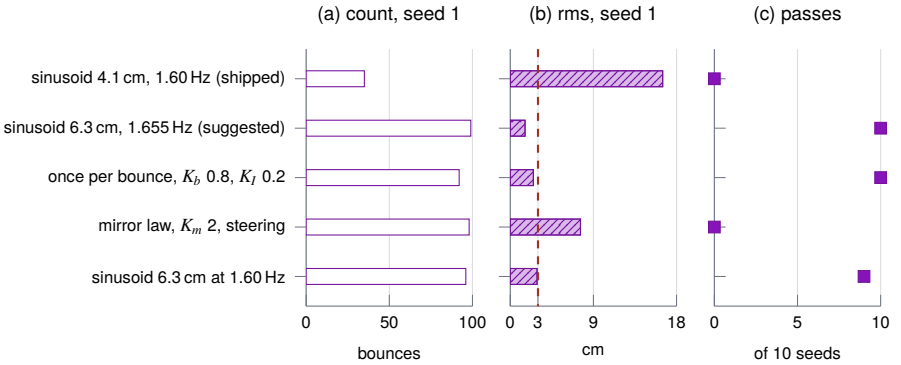


Figure 15.4 The challenge minute of Table 15.6 in its five configurations: (a) the count of consecutive bounces at seed 1 as white bars, (b) the apex error at seed 1 as shaded bars against the 3 cm rms target (dashed) and (c) the passes over ten seeds as dark squares. The table prints the values. Notice that the mirror law keeps almost as many bounces as the suggested sinusoid yet fails on its error, and that the once-per-bounce feedback passes with a larger error than the blind stroke.

Table 15.6 The Challenge Over 60 s, Seeds 1 to 10

Configuration	Count seed 1	Rms seed 1 (cm)	Rms, 10 seeds (cm)	Passes
sinusoid 4.1 cm, 1.60 Hz (shipped)	35[a]	16.5	16.508–21.450	0 of 10
sinusoid 6.3 cm, 1.655 Hz (suggested)	99	1.6	1.630–2.236	10 of 10
once per bounce, K_b 0.8, K_I 0.2	92	2.5	2.252–2.690	10 of 10
mirror law, K_m 2, steering	98	7.6	7.645–7.942	0 of 10
sinusoid 6.3 cm at 1.60 Hz	96	2.9	2.742–3.065	9 of 10

[a]Dropped at 23.2 s (23.1 to 23.4 s over the seeds, after 35 or 36 counted bounces).

15.7 Technology in Practice

What Measures It

Juggling robots have sensed the ball in three ways. The batting robot of 1989 read its puck through an oscillator inside it and a grid in its plane [14], a three-axis arm juggled two balls on a real-time stereo camera system [15], and the blind jugglers sense no more than their impacts [11, 12]. To judge a blind robot its builders still added instruments. A microphone timed each impact and lit a small array of lights that a video camera above the paddle could see, the ball's place on the paddle came from those frames in an offline analysis, and the motor encoder gave the paddle's position and speed at impact [12]. The impact switch and the arm's encoders of this chapter, and its camera, are these two kinds of measurement (Table 15.1).

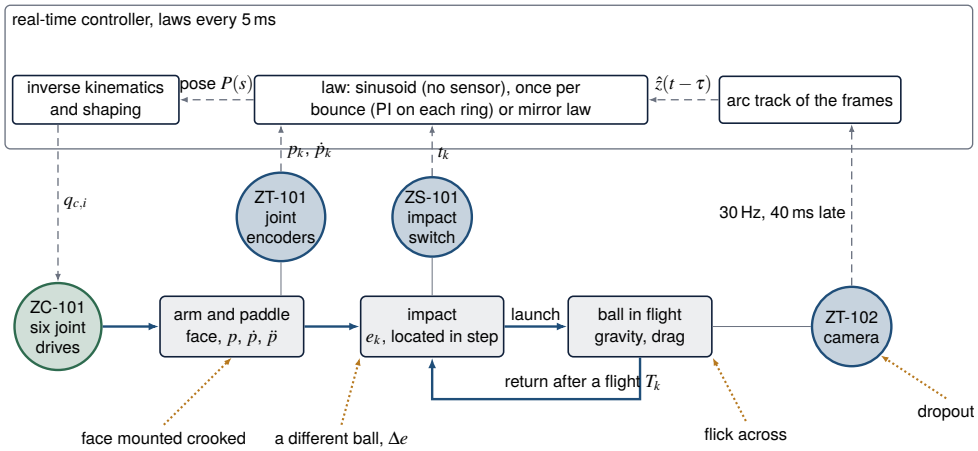


Figure 15.5 The juggling loop drawn on real hardware with the chapter's symbols. Sensors are drawn blue, the final element green and disturbances dotted. The impact switch, in practice a vibration sensor on the paddle, gives the impact times t_k, the joint encoders give the paddle's height and speed at each impact, and the camera's late frames feed the arc track. Only the mirror law reads the camera, only the once-per-bounce law the switch, and the sinusoid reads nothing.

> **TECHNOLOGY** **Sensor: Accelerometers and Microphones That Hear the Impact**
>
> A real impact switch is a vibration sensor. On one planar juggler two accelerometers at the tops of its arms recorded the small high-frequency vibrations of each impact, and their raw signals were band-pass filtered to remove the slow accelerations of the arms' own motion and the noise above [11]. A blind paddle robot heard its impacts through contact microphones on the paddle and read the paddle's position and speed at impact from its motor encoder [12, 16]. The sensor fails by missing an impact or by reporting a false one. Those microphones did not detect every impact, so a ball state was trusted only after a consistent series of impact times [16], and the planar juggler smoothed its arm motion to prevent false detections [11]. The chapter's switch rings at every impact.

What Moves It

A blind juggler can move its paddle along one axis. One such robot drove a concave aluminum paddle with an industrial linear motor, sized to juggle balls with a restitution of at least 0.7 to apex heights of at least 1.2 m, and a servo controller tracked the paddle's trajectory [12]. A juggling arm moves the paddle through all its joints at once. Figure 15.5 draws the loop of this chapter on such an arm, its drives modeled by Equation (15.4).

> **TECHNOLOGY** **Final Element: An Arm Driven Near Its Torque Limits**
>
> One learning system for a cable-driven juggling arm interpolated via-points in joint space with cubic splines and tracked them with a PD controller and gravity compensation, at accelerations of up to eight times gravity [17]. Tight box constraints on the joint trajectory kept the arm from hitting its joint limits. Rigid-body simulators could not represent the effects near the torque limits or the cable dynamics at high acceleration, and the wear ruled out running for days [17]. The arm of this chapter has speed and acceleration limits but no torque limits or flexibility.

Where the Controller Runs

The juggling controllers of the literature run on dedicated real-time computers. In the batting robot of 1989 the observer reported to the planning node at 1 kHz and the motor controller issued

torque commands at 2 kHz [14]. The planar blind juggler generated its code from a block-diagram simulation tool, ran it on a dedicated target computer under a real-time operating system, and sampled at 1 kHz [11]. The blind paddle robot split the work, a state machine on a real-time control system timing each stroke and passing profile parameters to the motor's servo controller [12]. The laws of this chapter run every 5 ms with the joint commands (Table 15.2), but the once-per-bounce law acts only when the switch rings, so its sample is the bounce.

> **IN PRACTICE** **Timing the Deceleration From Impact Statistics**
>
> A blind paddle robot was brought into service in steps. The servo controller of its linear motor was tuned by hand until the tracking was acceptable, and a real-time control system timed the stroke and gave the servo its profile parameters [12]. Around each nominal impact the paddle decelerated for a set interval, first sized for a static error in the restitution and then fixed from the statistics of the measured impact times. An interval of 0.05 s worked well, and a longer one added vertical robustness, because larger timing errors still met a decelerating paddle [12]. The once-per-bounce law of this chapter also keeps decelerating on its parabola when the ball is late.

What Is Modernizing It

Three developments touch this loop. The first is learning on the real arm. A cable-driven arm learned to juggle two balls from 56 minutes of experience with a binary reward, and its final policy juggled for up to 33 minutes, about 4500 repeated catches [17]. The second is learning a correction on top of an idealized plan. A residual learned from the task error gave stable three-, four- and five-ball juggling on two arms, open loop with respect to the balls, and converged from the second attempt [18]. The third is control that runs on events rather than on a clock. In event-triggered control the inputs are updated only when events occur, while a controller that executes periodically can load shared computing and communication more than the task needs [19]. The once-per-bounce law of Section 15.5 is a controller of that kind, with the impact as its event.

15.8 Limitations

The ball is a rigid, spinless sphere in one plane with a restitution that does not depend on speed, and its impacts are instantaneous and frictionless across the face, so spin, grip and the fall of restitution with impact speed are represented only by the bounce-to-bounce spread. A ball whose restitution jumps at one instant would not occur on a real paddle, but the step separates the margin of the stroke from the slow wear that a real ball shows. The face is flat, so a drifting ball is steered back only through the camera. The arm moves through exact kinematics and speed- and acceleration-limited drives with no rigid-body dynamics, and its stroke and tilt limits are teaching values chosen for wrist margin rather than taken from a particular robot.

> **OPEN CHALLENGE** **Trusting a Learned Juggler in the Loop**
>
> Closed-loop policies suit robot juggling, because the contacts between hand and ball are not consistent and feedback could recover from the irregularities. A learned closed-loop policy is hard to trust, however. Noisy ball observations or outliers might make it unstable and drive the arm into its joint limits, and its builders knew of no verification process for complex closed-loop networks, so they ran an open-loop stroke whose safety rests on the tracking controller and the joint constraints [17]. Progress would be a guarantee that a closed-loop policy is stable for every possible observation, edge cases included.

The analysis is local. Equation (15.7) describes small errors about the period-1 orbit, and the basin of that orbit, how far a bounce can stray and still return, was measured on the engine only through the phases of case 3. The once-per-bounce integral has no anti-windup, the mirror law's eigenvalue of $1 - K_m$ holds only for an ideal camera and paddle, and the camera has a fixed rate, latency and Gaussian noise with no blur or misread frames. Every threshold, the 5 mm/s at which a ball comes to

rest, the stroke and tilt limits and the challenge's 3 cm rms, is a teaching value of a simulator and not a measured property of a juggling robot.

15.9 Conclusion

A ball on a paddle is controlled one bounce at a time, and this chapter put that map on a six-axis arm with limited drives, an impact switch and a late camera. The linearized impact map has a determinant of e^2 for any stroke and a trace that falls with the paddle's acceleration at impact, so a paddle slowing between 0 and 9.93 m/s^2 holds the orbit with no sensor, and the eigenvalues measured with the arm in the loop, 0.79 to 0.83, match the closed form's 0.80. A 6.3 cm stroke at 1.655 Hz keeps the ball at 0.502 m within 1.6 cm and carries it blind through a change of ball with the apex 1 cm lower, while near either edge of the window the ball's own spread decides. The mirror law on a 40 ms camera settles 3 to 13 cm short as its gain falls, a camera 120 ms late defeats every gain, and a once-per-bounce integral removes the bias of a deader ball but lets in more spread than it removes. The model is planar, its impacts ideal and its arm without dynamics. A fuller model would add spin and a speed-dependent restitution, a concave face, rigid-body arm dynamics, and a basin-of-attraction map beside the eigenvalue window. The juggler is a machine whose stability is designed into its stroke before any sensor is added.

KEY POINTS

- A paddle that is slowing down when it meets the ball holds the bounce with no sensor, because a late ball gets a smaller push and comes back sooner.
- The linearized impact map has a determinant of e^2 for any stroke, and the orbit is stable for paddle accelerations at impact between −9.93 m/s^2 and zero.
- A 6.3 cm stroke at 1.655 Hz keeps the ball at 0.502 m within 1.6 cm and carries it blind through a change of ball.
- A camera 120 ms late defeats every mirror gain, and a once-per-bounce integral removes the bias of a deader ball but lets in more spread than it removes.

Where else this applies. A correction that waits for its measurements also runs the filling line of Chapter 9: with one group in four weighed instead of every group, the per-head correction at K 0.10 passes at none of 12 seeds instead of all 12 (Table 9.6).

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15.10 Exercises

15.1 Concept check · 2 min

The 6.3 cm sinusoid at 1.655 Hz keeps the ball bouncing with the camera off and reads no other sensor. Why does a bounce that comes back a little late not drift further off the beat?

(a) A late ball meets a paddle that has already slowed, gets a smaller push and comes back sooner

(b) A late ball meets a paddle that is still speeding up, gets a larger push and is sent back on the beat

(c) The impact switch reports the late impact, and the stroke shifts its phase to meet the next bounce

(d) The air drag takes more energy from a high bounce, so every apex returns to the target by itself

15.2 Concept check · 1 min

With the arm in the loop, the restitution exact and no drag, a sinusoid at 1.566 Hz is designed to meet the ball while the paddle is speeding up at 2 m/s^2. The closed form gives an eigenvalue of 1.977 there. What does the engine show?

(a) The ball is lost within a few bounces, as the eigenvalue predicts

(b) The ball settles on the same stroke's slowing branch, at about −2 m/s^2

(c) The orbit splits into a high and a low apex, as it does past the lower edge

(d) The ball holds the speeding-up orbit, because the servos add the damping it lacks

15.3 Concept check · 2 min

In Scenario 9 the mirror law drops a 0.20 m bounce through a camera 120 ms late at every gain from 0.25 to 2.5. Which change keeps the ball up?

(a) Add an integral term to the mirror gain, so the apex error is removed over several bounces

(b) Lower the target apex further, so each bounce needs less energy from the paddle

(c) Switch the camera off and run a 3 cm sinusoid at 2.5 Hz, which needs no measurement

(d) Raise the camera's frame rate from 30 to 120 Hz while its latency stays at 120 ms

15.4 Theory and calculation · 6 min

A drag-free ball with a constant restitution $e = 0.80$ is juggled to an apex $h = 0.50\,\mathrm{m}$ above the point of impact, with $g = 9.81\,\mathrm{m/s^2}$.

(a) Find the launch speed v, the flight time T and the upward paddle speed u^* at impact that hold the period-1 orbit (Equation (15.6)). *Checkpoint:* $v = 3.13\,\mathrm{m/s}$, $T = 0.639\,\mathrm{s}$, $u^* = 0.348\,\mathrm{m/s}$.

(b) The same ball is let go from rest 0.50 m above a still paddle. Each flight lasts e times the one before. Sum the fall and every later flight to find when the bounces end. *Checkpoint:* $0.319 + 2ev/\big(g(1-e)\big) = 2.87\,\mathrm{s}$.

(c) Case 2 measures an average apex ratio of 0.650 and a ball at rest after 29 impacts in 2.98 s. Repeat (b) with the effective restitution $\sqrt{0.650}$, and say what kind of solution of a hybrid system the still paddle shows. *Checkpoint:* $e = 0.806$ gives 2.97 s. A Zeno solution.

15.5 Theory and calculation · 7 min

Use the linearized impact map of Equation (15.7) with $g = 9.81\,\mathrm{m/s^2}$.

(a) At $e = 0.80$ and a paddle acceleration at impact $a = -5\,\mathrm{m/s^2}$, find $\mathrm{tr}\,J$ and the eigenvalues. How fast does an error shrink? *Checkpoint:* $\mathrm{tr}\,J = -0.011$, a complex pair of modulus 0.800.

(b) Repeat for $a = +2\,\mathrm{m/s^2}$. *Checkpoint:* $\mathrm{tr}\,J = 2.301$, eigenvalues 1.977 and 0.324.

(c) Find the lower edge of the window, Equation (15.9), for $e = 0.80$ and for a deader ball with $e = 0.72$. *Checkpoint:* $-9.931\,\mathrm{m/s^2}$ and $-10.070\,\mathrm{m/s^2}$.

15.6 Theory and calculation · 8 min

A sinusoidal stroke $z_p = A\sin\omega t$ meets a bounce lasting $1/f$, and on its slowing branch it meets the ball at the design acceleration of Equation (15.11). Take $g = 9.81\,\mathrm{m/s^2}$ and no drag.

(a) At $f = 1.655\,\mathrm{Hz}$, find the paddle speed u^* the bounce needs for $e = 0.80$ and for $e = 0.72$. *Checkpoint:* 0.329 m/s and 0.482 m/s.

(b) Find the design acceleration of the 6.3 cm stroke at 1.655 Hz for both balls. *Checkpoint:* $-5.89\,\mathrm{m/s^2}$ and $-4.61\,\mathrm{m/s^2}$.

(c) The challenge ships a 4.1 cm stroke at 1.60 Hz and changes the ball to $e = 0.72$ at 20 s. Compare the stroke's top speed with the speed the deader ball needs, and say what happens. *Checkpoint:* Top speed 0.412 m/s against 0.499 m/s: no impact can hold the orbit.

15.7 App challenge · 7 min

Open Scenario 2, "2 · A still paddle", at the default seed, and run at 1×. The run pauses by itself when the ball comes to rest.

(a) Run it as loaded. Read the first six apexes from the Orbit view's table, divide each by the one before (start from the 50 cm release), and average the six ratios. Estimate e. Tolerance 0.005 on the ratio. *Checkpoint:* 33.2, 22.1, 14.3, 9.2, 5.9, 3.8 cm. Mean ratio 0.650, $e \approx 0.806$.

(b) Read the clock and the impact count on the Sensors card when the run pauses. Then press Reset, set A different ball to -0.10 and run again. Read the same two values and the ratio of the second apex to the first. Tolerance 0.01 s. *Checkpoint:* 2.98 s and 29 impacts. Then 1.88 s, 19 impacts, ratio about 0.50.

15.8 App challenge · 8 min

Open Scenario 4, "4 · The window", at the default seed, and run at 1×. Each part sets Stroke amplitude before pressing Run.

(a) Set Stroke amplitude to 9.75 cm and run to 30 s. Read the Design acceleration at impact tile, compare it with the window's lower edge, and read the lowest and highest apex between 10 and

30 s on the Map view or in the exported column `truth_apex_m`. Tolerance 0.02 m. *Checkpoint:* $-10.0\,\mathrm{m/s^2}$, outside the edge at $-9.93\,\mathrm{m/s^2}$. Apexes from about 0.13 to 0.94 m.

(b) Reload the scenario, set Stroke amplitude to 3.3 cm and run. Read the tile and the time at which the ball comes to rest, then explain why a stroke inside the window loses the ball. Tolerance 0.1 s. *Checkpoint:* $-1.0\,\mathrm{m/s^2}$, dropped at 14.6 s after 21 counted bounces.

15.9 App challenge · 9 min

Open Scenario 7, "7 · Once per bounce", at the default seed, and run at 2× (two seconds of plant time per wall second). The ball is changed for one of restitution 0.72 at 12 s. In each part run to 30 s, export the CSV file and average the column `truth_apex_m` over the rows from 20 to 30 s. Tolerance 0.005 m.

(a) Run it as loaded, with Integral per bounce at 0. *Checkpoint:* 0.459 m.

(b) Press Reset, set Integral per bounce to 0.2 and run again. Also read the highest apex before 12 s. *Checkpoint:* 0.504 m, highest apex 0.569 m.

(c) Press Reset, set Bounce-to-bounce gain and Integral per bounce to 0, and run again. Explain the three results. *Checkpoint:* 0.383 m.

15.10 Application area · 3 min

An engineer measures a ball's restitution by dropping it from 0.50 m onto a still paddle and reading the apex ratios, as in case 2. Use Equation (15.1).

(a) Find the drag as a fraction of gravity at the 3.13 m/s of the first bounce, for the chapter's rubber ball with $k_D = 0.0075\,\mathrm{m^{-1}}$ and for a table-tennis ball with $k_D = 0.126\,\mathrm{m^{-1}}$. *Checkpoint:* 0.75 % and 12.6 %.

(b) What does that do to the measurement, and what should the engineer do? *Checkpoint:* The table-tennis ball reads e^2 up to 5.7 % low.

15.11 Application area · 4 min

A team builds a blind juggling machine for a trade-show booth, a paddle on a fixed sinusoid with no camera, to run all day with balls that wear. From the chapter's assumptions and limitations, name three things the real machine must handle that the simulator leaves out or idealizes, and say how the stroke should be chosen.

15.12 Review the agent · 5 min

For the challenge, an AI assistant writes: "In case 6 the mirror law at a gain of 2 holds the apex with a spread of 1.0 to 1.3 cm, tighter than the sinusoid's 1.6 cm. Set Law to Mirror law, Mirror gain to 2 and Paddle tilt law to Steer. The apex error will be about 1.3 cm rms, well inside the 3 cm target, and the camera feedback will also absorb the deader ball." Find the flaw and say how you would check the claim in the app.