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CHAPTER 9

A Riderless Benchmark Bicycle That Rides Itself Only Between Two Speeds

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CHAPTER 9

A Riderless Benchmark Bicycle That Rides Itself Only Between Two Speeds



The controller moves 1 steering motor that turns the front fork

The controller reads 1 rate gyro and accelerometer that sense the lean 2 encoder that reads the steer angle

What disturbs it 1 a side gust at the center of mass

Notice that the motor can only turn the front wheel: the frame stays up only if each steer brings the wheels back under its lean. Illustration, not to scale.

IN PLAIN TERMS A Bicycle That Steers Itself Upright

The main idea. A bicycle stays up because its front wheel steers under its fall. It is like a hand that moves under a stick balanced on a fingertip. With nobody on it, a bicycle does this by itself, but only inside a narrow band of speeds.

The challenge. Below that band it wobbles and falls. Above it the bicycle leans slowly into a spiral, so a motor on the handlebars must take over. Settings that hold it at one speed fail at another, and at a walking pace no controller tried can hold it up.

Why it matters. Aircraft, engines and process plants also change with speed or load, and their controllers must change with them.

THREE WAYS TO READ THIS CHAPTER

Explore	12 min	the first page, the history segments and the fact in Section 9.2, the try-its in Sections 9.4 and 9.6, the fact on trail in Section 9.6, and the key points
Study	45 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	15 min	the first page, the results (Section 9.6), Technology in Practice (Section 9.7), Limitations (Section 9.8) and the open challenge

The control question A riderless bicycle falls below 4.29 m/s, rides itself up to 6.02 m/s and slowly tips over above it. What does a speed-dependent plant ask of a lean loop, a schedule, a regulator, a rider and a path loop, and at what speed does no controller help?

Learning objectives

- Read the weave, capsize and castering modes and the self-stable window from the eigenvalues of the benchmark bicycle against speed.
- Compute the steer's grip on the lean, $gK_{0\varphi\delta} + v^2K_{2\varphi\delta}$, and explain why it rules out balance at walking pace.
- Explain why a balance loop must steer into the fall, and why a left turn starts with right steer.
- Compare a fixed lean PD, its two-gain schedule and a scheduled LQR across speed, counting the controller's sample hold.
- Order a path loop, a lean loop and a delayed rider by speed, and test each claim on the simulator.

Prerequisites Sections 1.4.3 (gain schedule), 1.4.4 (eigenvalues), 1.4.5 (LQR, Bryson's rule), 1.4.7 (RHP zero, delay) and 1.6.1 (engine). Chapter 7 for a cascade that commands an angle.

The app https://apmonitor.com/apps/bicycle-balance/

See also Volume 20, Chapter 2 for a path-following loop on a kinematic bicycle whose behavior changes with speed. Volume 3, Chapter 10 for an outer loop that commands an angle through a right-half-plane zero, and Volume 3, Chapter 9 for the bandwidth such a zero allows.

9.1 Introduction

A bicycle released at 3 m/s with its frame leaning 2° to the right is on the ground 2.00 s later. Released the same way at 5 m/s, it steers itself upright and the lean is below 0.5° by 5.5 s. At 8 m/s it leans slowly into a tightening spiral and passes 40° after 20.2 s. These three runs come from the simulator of this chapter, whose bicycle is the benchmark of Meijaard, Papadopoulos, Ruina and Schwab, stable with no rider and no control only between 4.292 and 6.024 m/s [1]. Nothing in the frame changes from one run to the next. Only the speed does.

Riding a bicycle teaches two beliefs that this simulator is designed to unsettle. The first is that stability is a property of a machine. The frame that rides itself at 5 m/s has a mode growing at 3.53 s^{-1} at 1 m/s, which doubles the lean every 0.20 s, and a lean loop tuned at 2 m/s is unstable on the same frame above 3.45 m/s. The second is that steering left turns a bicycle left. Asked for a 5° lean to the left at 5 m/s, the regulator in the simulator first steers 4.3° to the right, because a bicycle turns only once it leans and leans only once its wheels have run out from under it. A student reasoning from intuition may also get the sign of the balance loop backward, and with the wrong sign a lean loop drops the bicycle in 0.61 s, against 1.66 s with nobody steering at all.

The bicycle belongs to a broad class, plants whose stability and gain are set by an operating variable that the controller does not choose. The relation between a bicycle's design and its behavior depends heavily on speed [2], the stability of motorcycles has been compared across models that differ in their tire assumptions [3], and aircraft and engines are operated across their envelopes on gain schedules, one linear design for each operating condition [4]. A process unit whose throughput changes moves its own poles in the same way. Two further features of the bicycle are general. The response from steer to heading starts the wrong way, a right-half-plane zero that bounds what any loop wrapped around it can achieve [5, 6], and the authority of the steer over the lean depends on the speed, so that the controllability of the lateral motion is a question of its own [7].

The equations, the modes and the rider models of the bicycle are presented on paper [2, 8, 9], and an instrumented bicycle of the kind that tests them [10–12] falls over when a sign is wrong. This chapter shows what the self-stable window of the benchmark bicycle means for a lean loop, a schedule, a regulator, a rider and a path loop, on a browser simulator in which the eigenvalue map against speed, the running bicycle and the controller that holds it share one model. The equations, the benchmark values, the proportional-derivative (PD) and linear quadratic regulator (LQR) laws and

gain scheduling are all established. Section 9.2 follows the problem from the safety bicycle to the benchmark. Section 9.3 gives the model: matrices derived from 23 design parameters that equal the published ones to 1.4×10^{-14}, a nine-state model with a handlebar stop, a fall and a course, and a measurement boundary between the bicycle and its controller, so that the simulator can be rebuilt from this chapter. Section 9.4 reads the window in speed. Section 9.5 gives the controllers as coded and shows that the controller's 10 ms hold moves the stability edge of a lean PD tuned at 2 m/s from 5.59 m/s, where an analysis that applies the law continuously puts it, to 3.45 m/s, and that a two-gain schedule of the same loop leaves a band from 2.81 to 4.29 m/s in which the closed loop grows at up to 0.157 s^{-1}, while a scheduled regulator stays stable from 0.5 to 10 m/s. Section 9.6 reports the nine cases and the graded slalom as measured. Each is scored by a rule that credits completion, and no tuning can win case 9. Section 9.7 describes the sensors, steering motors and controllers of real test bicycles, Section 9.8 states the limitations, and the exercises of Section 9.10 work through the cases.

9.2 Self-Steering Bicycles, From the Safety Bicycle to the Benchmark

Balancing by steering is as old as the two-wheeler, and the history segment below follows it from the running machine to the first equations of a self-stable bicycle [1]. The model that carries Whipple's name, four rigid bodies on knife-edge wheels, is the one simulated here.

> ⧖ HISTORY **Balancing by Steering, 1818 to 1899**
>
> The review that accompanies the benchmark traces the idea. In 1818 Karl von Drais showed that a rider on two wheels in line could balance by steering the front wheel, and the pedal velocipede of the 1860s, with its vertical steering axis, was balanced by the rider's steering as well. A patent of 1866 stated the rule plainly: a carriage that leans to the right is righted by turning the wheel to the right. William Rankine's account of 1869 seems to be the first description of countersteering. By the 1890s the safety bicycle had a tilted steer axis and a bent fork, like a modern frame. In 1897 Emmanuel Carvallo and then, more generally, Francis Whipple in 1899 showed with rigid-body equations that some safety bicycles balance themselves in the right range of speed [1].

What makes a bicycle steer itself upright was argued for most of the following century. In 1970 Jones built bicycles from which selected features had been removed, among them one designed with computer calculations to be unstable [13], and for the bicycles he tried both the spin of the front wheel and a positive trail were needed [1]. Motorcycle models of the following year compared three assumptions about the tires, the stability characteristics each predicts and the effect of design changes [3]. By 2007 about 30 rigid-body models general enough to allow self-stability had been published, several of them in disagreement with the equations the benchmark then confirmed [1].

The benchmark of 2007 settled the Whipple model. Its authors derived the linearized equations by hand in two ways, checked them against two nonlinear simulation programs, and published one fully specified bicycle with its four matrices and its eigenvalues to fourteen digits so that any later formulation could be checked against them [1]. A companion paper wrote the nonlinear equations of the same bicycle and found three families of hands-free circular motions, two of them starting at the capsize bifurcation [14]. Measurements on a riderless bicycle whose masses and geometry had been measured found eigenvalues in good agreement with the formulas [1, 11], and in 2011 a bicycle built without gyroscopic or caster effects proved self-stable all the same, which points to the distribution of mass [12]. Case 3 of the simulator lets a student watch the benchmark's window move with the trail and the steer axis tilt.

DID YOU KNOW? Measuring the Window by Running Alongside

To test the model of an uncontrolled bicycle, a riderless bicycle with side wheels was launched 76 times in a university sports hall by a person running beside it. Most runs, 56, fell between 3.5 and 5 m/s, where the weave turns stable. Below 2 m/s the bicycle was so unstable that the time left to measure was very short [11].

Control came to the problem from two directions. Robotics asked for a bicycle that balances itself without a rider, on reduced models in which the steer is an input [1, 15], and found the balance problem nonminimum phase [6]. Control education took up bicycles adapted for teaching and research [10], and the same magazine surveyed bicycles, motorcycles and their models for the control community [8]. Rider control has been studied with optimal preview, which finds a rider's steering powerful compared with the bicycle's own self-steering, especially at low speed [2]. The stability and controllability of a bicycle that carries a passive rider model have been analyzed [7], and two reviews of 2013 collect the dynamics and the rider models [9, 16]. The rider model of case 8 has passive arms on the bars for a measured reason, given in Section 9.5.4.

HISTORY A Steering Motor Takes the Rider's Place, 1975 to 1986

Attempts to balance single-track vehicles automatically began once predictive motorcycle models had been refined in the 1970s. A thesis of 1975 tried to stabilize a small unmanned motorbike with a controlled pendulum that mimicked a rider's lean. It failed in practice, but it recognized that feedback of the roll angle and its rate was essential. In the early 1980s a robotic bicycle was balanced on a treadmill with steering control and a laterally moving mass. A study of 1986 then balanced a motorcycle with a steering motor alone. Roll angle feedback stabilized the capsize, roll rate feedback the weave and steer rate feedback the wobble, and the gains had to change with speed [17]. The scheduled laws of Section 9.5 make the same change.

9.3 The Bicycle as Modeled

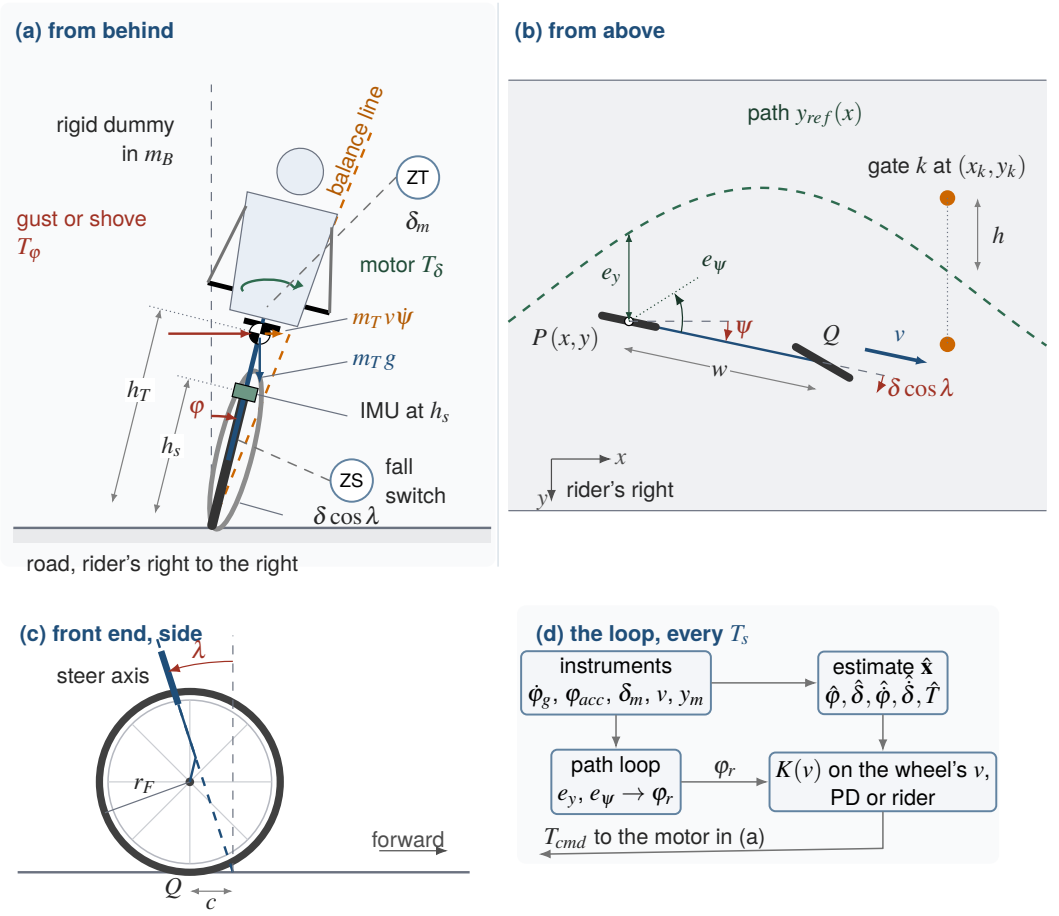
The simulated machine is the benchmark bicycle with a rigid dummy in the rider's place, a steer motor on the head tube, an inertial sensor on the frame, a steer encoder, a wheel-speed encoder, a lateral position sensor referenced to the lane and a tilt switch, as shown in Figure 9.1. Lean φ and steer δ are positive to the right, the heading ψ is positive clockwise seen from above, and (x, y) is the rear contact point with x along the road and y to the rider's right, the SAE J670e convention of the benchmark [1]. Every dimension and mass is the benchmark's, published for exactly this kind of reproduction. The motor, the instruments, the limits and every threshold are teaching values, and the application says so in its model dialog and on its Instructions page. Table 9.1 sorts the signals by role.

9.3.1 Lean and Steer

The Whipple bicycle is four rigid, laterally symmetric bodies hinged together, a rear wheel R, a rear frame B that includes the rider's body, a front frame H and a front wheel F, on knife-edge wheels that roll without slip on level ground. Linearized about upright, straight running at forward speed v, its lean and steer obey Equation (9.1) [1].

$$\mathbf{M}\ddot{\mathbf{q}} + v\,\mathbf{C}_1\dot{\mathbf{q}} + \left(g\,\mathbf{K}_0 + v^2\mathbf{K}_2\right)\mathbf{q} = \begin{bmatrix} T_\varphi \\ T_\delta \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} \varphi \\ \delta \end{bmatrix} \tag{9.1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}_1$ collects the gyroscopic and inertial terms that grow with v, $\mathbf{K}_0$ the gravity terms and $\mathbf{K}_2$ the terms that grow with v^2. Every entry is a formula in the design parameters of Table 9.2, which the simulator evaluates at every stage of every step (memoized, since the parameters change only when a student changes them). The bicycle's totals about the rear contact come first, as



IMU: gyro $\dot{\varphi}_g$, accelerometer φ_{acc} Motor: lag τ_a, $|T_\delta| \le T_{max}$ Fall: $|\varphi| \ge \varphi_{fall}$

Figure 9.1 The riderless benchmark bicycle as the Bike view draws it, and the loop that holds it, not to scale. (a) From behind, the frame leans φ about the contact line with the dummy in the rear body, and the center of mass at h_T carries the weight $m_T g$ and the centripetal force $m_T v\dot{\psi}$, whose resultant sets the balance line. A gust F at h_T or a shove at the saddle is the lean torque T_φ, the steer motor applies T_δ through the lag τ_a within $\pm T_{max}$, the front wheel turns out at the ground steer angle $\delta\cos\lambda$, the IMU at h_s carries the gyro and the accelerometer, the encoder ZT reads δ_m, and the tilt switch ZS cuts the motor at φ_{fall}. (b) From above, the rear contact P at (x,y) heads at ψ with speed v, the front contact Q is the wheelbase w ahead, and the path loop sees the lateral error e_y and the heading error e_ψ against a path through gates of half-width h. (c) The front end from the side, with the trail c, the steer axis tilt λ and the wheel radius r_F, as the Map view draws it. (d) Every T_s the controller estimates the state from the instruments and applies a law scheduled on the wheel's speed.

Table 9.1 Signals of the Bicycle by Role

Symbol	Role	Nominal	Range
φ	CV and state, lean, right positive	per case	FALLEN at $\pm 40°$
φ_r	SP, lean setpoint or the path loop's output	0	$\pm 15°$
δ	state, steer, right positive	0	stop at $\pm 45°$
$\dot{\varphi}$, $\dot{\delta}$	states, lean and steer rates	0	—
T_δ	state, steer motor torque	0	± 10 N m
T_{cmd}	MV, torque command	per law	$\pm T_{max}$
v	state, forward speed	per case	SP 0 to 10 m/s
ψ, x, y	states, heading and rear contact	0	—
$\dot{\varphi}_g$	measured, rate gyro	$\dot{\varphi}$	noise 0.1 °/s
φ_{acc}	measured, accelerometer tilt	φ	noise 0.5°
δ_m	measured, steer encoder	δ	2^{14} counts a turn
y_m, ψ_m	measured, position and heading	y, ψ	0.02 m, 0.2°
T_φ	DV, gust, side wind, shove, cross slope	0	per event
J_p	DV, pothole, a steer impulse	—	0.5 N m s
m_p	DV, passenger on the rear rack	0	0 to 40 kg
b_g	DV, gyro bias	0	± 5 °/s

shown in Equation (9.2).

$$\begin{aligned}
m_T &= m_R + m_B + m_H + m_F \\
x_T &= (x_B m_B + x_H m_H + w m_F)/m_T \\
z_T &= (-r_R m_R + z_B m_B + z_H m_H - r_F m_F)/m_T \\
I_{Txx} &= I_{Rxx} + I_{Bxx} + I_{Hxx} + I_{Fxx} \\
&\quad + m_R r_R^2 + m_B z_B^2 + m_H z_H^2 + m_F r_F^2 \\
I_{Txz} &= I_{Bxz} + I_{Hxz} - m_B x_B z_B - m_H x_H z_H + m_F w r_F \\
I_{Tzz} &= I_{Rxx} + I_{Bzz} + I_{Hzz} + I_{Fxx} \\
&\quad + m_B x_B^2 + m_H x_H^2 + m_F w^2
\end{aligned} \tag{9.2}$$

where the wheels are axisymmetric, so $I_{Rzz} = I_{Rxx}$ and $I_{Fzz} = I_{Fxx}$, and heights z are negative because the z axis points down. The front assembly A, the front frame with its wheel, has its own totals about its center of mass, as shown in Equation (9.3).

$$\begin{aligned}
m_A &= m_H + m_F, \quad x_A = (x_H m_H + w m_F)/m_A \\
z_A &= (z_H m_H - r_F m_F)/m_A \\
I_{Axx} &= I_{Hxx} + I_{Fxx} + m_H (z_H - z_A)^2 + m_F (r_F + z_A)^2 \\
I_{Axz} &= I_{Hxz} - m_H (x_H - x_A)(z_H - z_A) \\
&\quad + m_F (w - x_A)(r_F + z_A) \\
I_{Azz} &= I_{Hzz} + I_{Fxx} + m_H (x_H - x_A)^2 + m_F (w - x_A)^2 \\
u_A &= (x_A - w - c)\cos\lambda - z_A \sin\lambda \\
I_{A\lambda\lambda} &= m_A u_A^2 + I_{Axx} \sin^2\lambda \\
&\quad + 2 I_{Axz} \sin\lambda \cos\lambda + I_{Azz} \cos^2\lambda \\
I_{A\lambda x} &= -m_A u_A z_A + I_{Axx} \sin\lambda + I_{Axz} \cos\lambda \\
I_{A\lambda z} &= m_A u_A x_A + I_{Axz} \sin\lambda + I_{Azz} \cos\lambda
\end{aligned} \tag{9.3}$$

where u_A is the distance of the assembly's center of mass ahead of the steer axis, $I_{A\lambda\lambda}$ its moment of inertia about that axis, and $I_{A\lambda x}$ and $I_{A\lambda z}$ products about crossed axes. With the trail ratio $\mu =$

Table 9.2 Model Parameters. B Is the Benchmark's Published Value, A a Teaching Value Adjustable in the Model Dialog, H a Hidden Teaching Value

Symbol	Meaning	Value	Range	
w	wheelbase	1.02 m	0.6–1.6	B
c	trail	0.08 m	−0.1–0.2	B
λ	steer axis tilt	$\pi/10$ (18°)	0–40°	B
g	gravity	9.81 m/s^2	1.6–24.8	B
r_R, m_R	rear wheel	0.3, 2		B
I_{Rxx}, I_{Ryy}	its inertias	0.0603, 0.12		B
x_B, z_B, m_B	rear frame, dummy	0.3, −0.9, 85		B
I_{Bxx}, I_{Bxz}, I_{Bzz}	its inertias	9.2, 2.4, 2.8		B
x_H, z_H, m_H	front frame	0.9, −0.7, 4		B
I_{Hxx}, I_{Hxz}, I_{Hzz}	its inertias	0.05892, −0.00756, 0.00708		B
r_F, m_F	front wheel	0.35, 3		B
I_{Fxx}, I_{Fyy}	its inertias	0.1405, 0.28		B
T_{max}	steer motor limit	10 N m	1–30	A
τ_a	steer motor lag	20 ms	2–200	A
τ_v	drive speed loop	0.5 s	0.1–5	A
σ_g	gyro noise	0.1 °/s	0–3	A
σ_a	accelerometer noise	0.5°	0–5	A
h_s	sensor height	0.6 m	0.1–1.2	A
N_δ	encoder resolution	14 bit	8–18	A
σ_y	position noise	0.02 m	0–0.5	A
φ_{fall}, δ_{lock}	fall angle, stop	40°, 45°	fixed	H

Lengths in m, masses in kg and inertias in kg m^2. Every benchmark value is adjustable in the model dialog. The frames' inertias about the lateral axis, $I_{Byy} = 11\,\mathrm{kg\,m^2}$ and $I_{Hyy} = 0.06\,\mathrm{kg\,m^2}$, complete the benchmark's 25 parameters but do not enter Equations (9.2) to (9.4), so the simulator omits them.

$(c/w)\cos\lambda$, the gyrostatic coefficients $S_R = I_{Ryy}/r_R$, $S_F = I_{Fyy}/r_F$ and $S_T = S_R + S_F$, and the static moment $S_A = m_A u_A + \mu m_T x_T$, the entries of the matrices are Equation (9.4).

$$
\begin{aligned}
M_{\varphi\varphi} &= I_{Txx}, \quad M_{\varphi\delta} = M_{\delta\varphi} = I_{A\lambda x} + \mu I_{Txz} \\
M_{\delta\delta} &= I_{A\lambda\lambda} + 2\mu I_{A\lambda z} + \mu^2 I_{Tzz} \\
K_{0\varphi\varphi} &= m_T z_T, \quad K_{0\varphi\delta} = K_{0\delta\varphi} = -S_A \\
K_{0\delta\delta} &= -S_A \sin\lambda \\
K_{2\varphi\delta} &= (S_T - m_T z_T)\cos\lambda/w, \quad K_{2\varphi\varphi} = K_{2\delta\varphi} = 0 \\
K_{2\delta\delta} &= (S_A + S_F \sin\lambda)\cos\lambda/w \\
C_{1\varphi\delta} &= \mu S_T + S_F \cos\lambda + I_{Txz}\cos\lambda/w - \mu m_T z_T \\
C_{1\delta\varphi} &= -(\mu S_T + S_F \cos\lambda), \quad C_{1\varphi\varphi} = 0 \\
C_{1\delta\delta} &= I_{A\lambda z}\cos\lambda/w + \mu\left(S_A + I_{Tzz}\cos\lambda/w\right)
\end{aligned}
\tag{9.4}
$$

These are the formulas of the benchmark's appendix [1]. For the benchmark bicycle they give $m_T = 94\,\mathrm{kg}$, $x_T = 0.342\,\mathrm{m}$, $z_T = -0.861\,\mathrm{m}$, $\mu = 0.0746$ and $S_A = 2.600\,\mathrm{kg\,m}$. A passenger of mass m_p on the rear rack, at $x = -0.05\,\mathrm{m}$ and $z = -1.0\,\mathrm{m}$, is folded into the rear frame by the parallel-axis theorem before the derivation, so it changes every matrix the physics integrates, while the controller keeps the matrices of the bicycle it was told about.

Table 9.3 Eigenvalues of the Benchmark Bicycle From the Simulator's Own Derivation (1/s), and the Largest Difference From the Published Table in Each Row

v (m/s)	Re weave	Im weave	capsize	castering	max $\lvert\Delta\rvert$
0	real: 5.530944, 3.131643	−3.131643	−5.530944	5.3×10^{-15}	
1	3.526962	0.807740	−3.134231	−7.110080	1.4×10^{-14}
2	2.682345	1.680663	−3.071586	−8.673880	1.4×10^{-14}
3	1.706756	2.315824	−2.633661	−10.351015	2.5×10^{-14}
4	0.413253	3.079108	−1.429444	−12.158614	4.3×10^{-14}
5	−0.775342	4.464868	−0.322866	−14.078390	1.6×10^{-14}
6	−1.526445	5.876731	−0.004067	−16.085371	1.1×10^{-14}
7	−2.138756	7.195259	0.102682	−18.157885	6.1×10^{-13}
8	−2.693487	8.460380	0.143279	−20.279409	5.3×10^{-14}
9	−3.216754	9.693774	0.157902	−22.437886	8.3×10^{-14}
10	−3.720168	10.906811	0.161053	−24.624596	7.5×10^{-14}

9.3.2 The Benchmark Reproduced

With the parameters of Table 9.2, the sixteen derived entries equal the matrices printed in the benchmark [1], Equation (9.5), to 1.4×10^{-14}, the largest difference being in the entry -80.95.

$$\begin{aligned}
\mathbf{M} &= \begin{bmatrix} 80.81722 & 2.31941332208709 \\ 2.31941332208709 & 0.29784188199686 \end{bmatrix} \\
\mathbf{K}_0 &= \begin{bmatrix} -80.95 & -2.59951685249872 \\ -2.59951685249872 & -0.80329488458618 \end{bmatrix} \\
\mathbf{K}_2 &= \begin{bmatrix} 0 & 76.59734589573222 \\ 0 & 2.65431523794604 \end{bmatrix} \\
\mathbf{C}_1 &= \begin{bmatrix} 0 & 33.86641391492494 \\ -0.85035641456978 & 1.68540397397560 \end{bmatrix}
\end{aligned} \tag{9.5}$$

The eigenvalues are taken from the state matrix $\mathbf{A}(v) = [\mathbf{0},\ \mathbf{I};\ -\mathbf{M}^{-1}(g\mathbf{K}_0 + v^2\mathbf{K}_2),\ -v\mathbf{M}^{-1}\mathbf{C}_1]$ by the balanced Hessenberg QR routine of the shared framework (Section 1.6), `shared/src/framework/core/eig.js`. Against the 44 numbers of the benchmark's eigenvalue table from 0 to 10 m/s, the real and imaginary parts of the weave, the capsize and castering roots and the roots at rest, they agree to 6.2×10^{-13}, and 43 of them to 8.4×10^{-14}, as listed in Table 9.3. The exception is the castering root at 7 m/s, -18.15788466125201 against the printed -18.15788466125262. Newton's method on the characteristic quartic of the derived matrices returns the simulator's value to the last bit, so the difference lies in the thirteenth decimal of the printed value rather than in the derivation. Bisection on the largest real part finds the weave speed $v_w = 4.292\,382\,536\,341\,10\,\mathrm{m/s}$ and the capsize speed $v_c = 6.024\,262\,015\,388\,36\,\mathrm{m/s}$, against the printed 4.29238253634111 and 6.02426201538837, differences of 5.3×10^{-15} and 8.9×10^{-15}. The weave frequency at v_w, $3.435\,033\,848\,661\,44\,\mathrm{s^{-1}}$, and the speed $v_d = 0.684\,283\,078\,892\,\mathrm{m/s}$ at which the weave pair forms both agree to 3.2×10^{-15}. The published values were transcribed from the benchmark's preprint (final revision of 9 June 2007), because the publisher's site refused automated requests.

9.3.3 Path, Drive and Steer Motor

The rear contact moves along its heading, and the heading turns with the steer by the benchmark's kinematics [1], as shown in Equation (9.6).

$$\dot{\psi} = \frac{(v\delta + c\dot{\delta})\cos\lambda}{w}, \qquad \dot{x} = v\cos\psi, \qquad \dot{y} = v\sin\psi \tag{9.6}$$

A first-order speed loop on the rear wheel, limited to the drive's acceleration, holds the forward speed, and the steer motor's torque lags its command, as shown in Equation (9.7).

$$\begin{aligned} \dot{v} &= \mathrm{sat}_{\pm a_{max}}\left(\frac{v_{SP}-v}{\tau_v}\right) \\ \tau_a \dot{T}_\delta &= T_{cmd} - T_\delta, \qquad |T_{cmd}| \le T_{max} \end{aligned} \tag{9.7}$$

where $\tau_v = 0.5\,\mathrm{s}$, $a_{max} = 1.5\,\mathrm{m/s^2}$ by default, $\tau_a = 20\,\mathrm{ms}$ and $T_{max} = 10\,\mathrm{N\,m}$, and the speed is held at or above zero. The benchmark is linearized at constant speed, and here the matrices of Equation (9.1) simply follow v as it changes, a quasi-steady simplification that the benchmark does not cover.

9.3.4 Fall, Stop and Disturbances

Three hard events act between integration steps. FALLEN latches where $|\varphi|$ crosses $\varphi_{fall} = 40^\circ$, the instant located by linear interpolation inside the step, and the tilt switch then cuts the motor while the bicycle topples about its contact line as one rigid body, $\ddot{\varphi} = m_T g h_T \sin\varphi / I_{Txx}$ with $h_T = -z_T$, until the bars touch the ground at 72°, where the run stops. The handlebar stop at $\delta_{lock} = \pm 45^\circ$ is an inelastic, one-sided constraint. On arrival the steer rate toward the stop is removed by an impulse J on the steer coordinate, which reaches the lean through the mass matrix as $\Delta\dot{\mathbf{q}} = \mathbf{M}^{-1}[0,\ J]^{\mathsf{T}}$, and while the steer is held only the lean row of Equation (9.1) is integrated, until the free steer acceleration points away from the stop. A pothole is the same kind of impulse, $J_p = 0.5\,\mathrm{N\,m\,s}$ on the steer. The lean torque is Equation (9.8).

$$T_\varphi = F h_T + m_T g h_T \arctan(s_r/100) + T_{sh} \tag{9.8}$$

where F is a side wind of up to $\pm 100\,\mathrm{N}$ plus any gust (70 N for 1.5 s) at the center of mass, s_r is a road cross slope in percent, and T_{sh} is a shove of 50 N at the saddle, 0.9 m up, for 0.4 s. Every one of these sizes is a teaching value.

9.3.5 Instruments

The controller reads the bicycle only through a measurement packet, assembled once every control period $T_s = 10\,\mathrm{ms}$ by one function, as shown in Equation (9.9).

$$\begin{aligned} \dot{\varphi}_g &= \dot{\varphi} + b_g + \sigma_g n_1, \qquad \varphi_{acc} = \varphi - \frac{v\dot{\psi} + h_s\ddot{\varphi}}{g} + \sigma_a n_2 \\ \delta_m &= q\,\mathrm{rnd}(\delta/q), \qquad q = 2\pi/2^{N_\delta} \\ y_m &= y + \sigma_y n_3, \qquad \psi_m = \psi + \sigma_\psi n_4 \end{aligned} \tag{9.9}$$

where n_1 to n_4 are unit normal deviates from the engine's generator seeded at 20261019, drawn on every call whatever the noise levels so that one changed noise does not reshuffle the others, $\ddot{\varphi}$ is the true lean rate differenced over the period, $\sigma_\psi = 0.2^\circ$, and the remaining values are in Table 9.2. An accelerometer cannot tell gravity from the bicycle's own sideways acceleration, so in a balanced turn φ_{acc} reads almost no lean. The wheel speed and the distance along the course are exact, and the tilt switch reports the fall.

9.3.6 Numerics and Initial States

Table 9.2 lists the parameters. The nine states, the lean and steer with their rates, the motor torque, the speed, the heading and the rear contact, are integrated by the engine of Section 1.6.1 in steps of 1 ms, against a 20 ms motor lag and a castering root of $-24.6\,\mathrm{s^{-1}}$ at 10 m/s. The controller runs every 10 ms and the historian logs at 50 Hz, which gives 29 samples per period of the fastest motion the trends show, the weave at 1.7 Hz at 10 m/s. A case starts upright and straight at its speed setpoint, or with the 2° lean its scenario names, the motor torque at zero. The noise is seeded, so every run replays exactly, and Reset and the reload button behave as Section 1.6.1 describes. The application passes its

49 verification checks. Among them the engine's integrated motion equals the matrix exponential of $\mathbf{A}(v)$ to 10^{-8} (relative) over 1 s at 1, 3, 5 and 8 m/s, and halving the step moves the countersteer peak of case 6 by 6×10^{-9} degree and the fall of case 1 by 0.4 ms. On the machine used here the model runs about 348 000 steps a second outside the browser, against the 5 000 a second that the top speed of 5× real time needs. At that speed the 40 s slalom takes 8 s to watch.

9.3.7 Assumptions

The Whipple bicycle. Rigid parts, ideal hinges and knife-edge wheels that roll without slip, with no tire compliance or relaxation, no frame flex, no friction in the steering head and no rider moving on the frame. Wobble, the fast steering oscillation of real machines, appears to need tire and frame compliance and is absent [1].

Linear about upright running. The lean is trusted for small angles, and FALLEN at 40° is a device for the picture. An optional nonlinear mode puts $\sin\varphi$ in the gravity term of the lean equation and nothing else, and with it on, above the capsize speed, the bicycle settles into a steady turn at 15 to 31° of lean from 6.5 to 10 m/s instead of falling, because the turn's terms stay linear. The full nonlinear benchmark has families of steady hands-free circles of its own [14], and the mode is not a model of them.

A rigid dummy. The rider's 85 kg belong to the rear frame and the dummy's hands rest on the bars without torque. Case 8's rider is a control law.

A stand-in for the road. A cross slope puts no lean torque on knife-edge wheels to first order, because the contact stays under the wheel plane, so the term in Equation (9.8) is a teaching stand-in for what real tires do. The drive has no pitch dynamics, rolling resistance or air drag.

Teaching limits. The 10 N m motor, the 20 ms lag, the 45° stop and the 40° fall angle are teaching values. A real steer motor's limits come from its data sheet and gearing, a real bicycle's steering lock from where its bars or fork crown meet the frame, and a self-balancing test bicycle cuts its motor at a tilt its builder chooses.

9.4 A Window in Speed

9.4.1 Four Modes Against Speed

At rest the bicycle is two inverted pendulums, with real roots at ± 3.13 and $\pm 5.53\ \mathrm{s}^{-1}$, as shown in Figure 9.2a. As the speed rises, the two positive roots meet at $v_d = 0.684\,\mathrm{m/s}$ and form the weave, an oscillation of lean and steer, while the negative ones become the capsize, a slow real mode dominated by lean, and the castering, a fast real mode dominated by steer [1]. The weave grows below $v_w = 4.292\,\mathrm{m/s}$ and the capsize above $v_c = 6.024\,\mathrm{m/s}$. Between the two every mode decays, the slowest at $0.32\ \mathrm{s}^{-1}$ at 5 m/s, and the bicycle steers itself into its own fall hard enough to catch it and not so hard that it holds itself in a turn. Hands off from the 2° push, it falls in 1.46 s at 1 m/s, where the weave grows at $3.53\ \mathrm{s}^{-1}$ with a period of 7.8 s, and in 2.00 s at 3 m/s, where the weave grows at $1.71\ \mathrm{s}^{-1}$ with a period of 2.7 s and the lean swings once to the other side before the fall. At 8 m/s the capsize grows at $0.143\ \mathrm{s}^{-1}$, an e-folding time of 7.0 s, and the lean reaches 40° after 20.2 s.

Pause. Try Exercise 9.1 before reading on.

> **TRY IT IN THE APP** **2 · The Window: Hands Off at 3, 5 and 8 m/s**
>
> Load the scenario and press Run with your hands off, and watch the lean. At 3 m/s it swings once to the other side and the bicycle is down at 2.00 s. Set Speed setpoint to 5 m/s, press Reset and run again: the bicycle rides upright. At 8 m/s it leans slowly and passes 40° after 20.2 s.

9.4.2 Where the Steer Gets Its Grip

The steer acts on the lean through the off-diagonal stiffness. Holding the steer at an angle δ puts a static torque $(gK_{0\varphi\delta} + v^2K_{2\varphi\delta})\delta$ on the lean, −25.5 N m per radian from gravity and a term growing

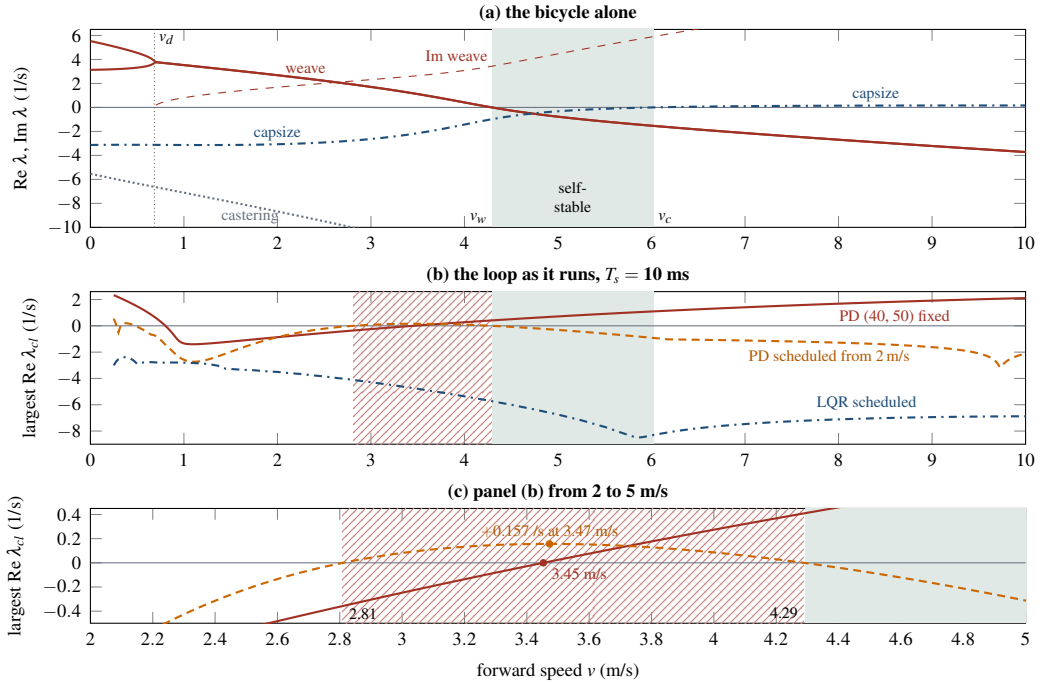


Figure 9.2 The eigenvalue map, from the simulator's own functions. (a) Real parts of the four eigenvalues of the benchmark bicycle against speed, with the weave's imaginary part dashed and the self-stable window from 4.292 to 6.024 m/s shaded. The weave's real parts are solid, the capsize dash-dotted and the castering dotted. (b) The largest real part of the closed loop as it runs, the bicycle held over each 10 ms period, for case 5's lean PD tuned at 2 m/s, fixed (solid) and scheduled (dashed), and for the scheduled LQR (dash-dotted). (c) The same from 2 to 5 m/s. The fixed PD loses stability at 3.45 m/s, and the schedule leaves a band from 2.81 to 4.29 m/s (hatched) in which the closed loop grows at up to 0.157 s^{-1}.

with the square of the speed, so that the grip is 1889 N m/rad at 5 m/s, 281 at 2 m/s, 51 at 1 m/s, zero at 0.58 m/s and reversed below. At 0.3 m/s it is −18.6 N m/rad, against −794 N m/rad of gravity on the lean, $gm_T z_T$. A lean loop's gain must therefore grow as the speed falls, and at a walking pace no gain the motor can deliver is enough. Case 9 is built on this fact. Below the zero the steer's gain on the lean has changed sign, as the gain of a braking tire does past its friction peak in Chapter 4.

🕮 **Pause.** Try Exercise 9.3 before reading on.

9.4.3 A Response That Starts the Wrong Way

Steering to the right turns the front contact to the right under a bicycle whose center of mass carries on, so the frame begins to lean to the left, and only a bicycle leaning left can turn left without falling over. From a lean or heading command to the path, the response therefore starts the wrong way, the signature of a right-half-plane zero [5, 6] (Section 1.4.7). In the linear model at 5 m/s the regulator of Section 9.5, asked for a 5° lean to the left, steers 4.0° to the right before 4.0° to the left, and the rear wheel's track moves 5.9 cm to the right before the path bends left. Any loop wrapped around the lean loop, such as the path loop of case 7, has to be slower than the lean loop for this reason. The cart loop of Chapter 7 and the position loop of Volume 3, Chapter 10, which command an angle through a wrong-way response, obey the same rule.

🕮 **Pause.** Try Exercise 9.2 before reading on.

Table 9.4 Controller Settings, Defaults and Ranges

Symbol	Setting	Default	Range
—	law	LQR	PD, rider
ε	PD sign	into the fall	wrong way
k_φ, $k_\dot{\varphi}$	PD gains (N m/rad, N m s/rad)	40, 50	0–200, 0–120
—	schedule on speed	on	off
v_d	design speed	5 m/s	0.5–10
$\bar{\varphi}$, $\bar{\delta}$, $\bar{T}$	LQR scales	2°, 5°, 5 N m	0.5–10, 1–30, 0.5–20
τ_{cf}	lean filter time constant	0.5 s	0.05–5
—	lean estimate	compensated	raw, perfect
k_r, T_d	rider reflex (N m/rad), anticipation	10, 0.4 s	0–100, 0–2
k_a, c_a	rider's arms (N m/rad, N m s/rad)	10, 0.5	0–60, 0–10
τ_r	rider's reaction	150 ms	0–400
ω_p, ζ_p	path loop	0.6 rad/s, 0.7	0.2–3, 0.3–2
t_p	curvature preview	0.4 s	0–2
v_{SP}, a_{max}	speed setpoint, drive limit	5 m/s, 1.5 m/s^2	0–10, 0.2–4

9.5 The Controller as Coded

Every 10 ms the controller reads the packet of Equation (9.9), updates its estimate, computes the lean reference, runs one law and writes a torque command clamped to $\pm T_{max}$ and held until the next period. Its own model is the declared bicycle, the parameters as set with no passenger, and no law reads the simulator's truth, which the export logs in columns prefixed `truth_`. Table 9.4 lists the settings.

9.5.1 Estimate

The laws act on $\hat{\mathbf{x}} = [\hat{\varphi}, \hat{\delta}, \hat{\dot{\varphi}}, \hat{\dot{\delta}}, \hat{T}]$. The lean is a complementary filter that integrates the gyro and pulls toward the accelerometer, corrected for the turn the encoder and the wheel speed imply and for the sensor's own swing, as shown in Equation (9.10).

$$\begin{aligned} \hat{\varphi}_k &= a\left(\hat{\varphi}_{k-1} + \dot{\varphi}_{g,k} T_s\right) + (1-a)\left(\varphi_{acc,k} + \frac{v\dot{\psi}_{kin} + h_s \hat{\dot{\varphi}}}{g}\right) \\ a &= \frac{\tau_{cf}}{\tau_{cf} + T_s}, \qquad \dot{\psi}_{kin} = \frac{(v\hat{\delta} + c\hat{\dot{\delta}})\cos\lambda}{w} \end{aligned} \tag{9.10}$$

where $\hat{\dot{\varphi}}$ is the gyro differenced and passed through a 20 ms first-order filter. The lean rate is the gyro as read, bias included, the steer is the encoder, the steer rate is the encoder differenced through the same 20 ms filter, and $\hat{T}$ is a replica of the motor's lag driven by the commands sent. Two options replace the filter, the raw accelerometer without the correction and the true state, which is labeled a teaching comparison.

9.5.2 The Lean PD and Its Schedule

The lean PD is Equation (9.11).

$$T_{cmd} = \varepsilon\left(k_\varphi(\hat{\varphi} - \varphi_r) + k_\dot{\varphi}\hat{\dot{\varphi}}\right) \tag{9.11}$$

with $\varepsilon = +1$ steering into the fall, a positive torque for a lean to the right, and $\varepsilon = -1$ the wrong way. On the lag-free bicycle the closed loop's characteristic polynomial is affine in the two gains, $p(s) = \det\left(\mathbf{M}s^2 + v\mathbf{C}_1 s + g\mathbf{K}_0 + v^2\mathbf{K}_2 - \mathbf{e}_2[k_\varphi + k_\dot{\varphi}s,\ 0]\right)$ with $\mathbf{e}_2 = [0,1]^\mathsf{T}$, and two gains cannot place its four roots. A schedule in the sense of Section 1.4.3, which Chapter 3 applies on angle, keeps the closed-loop polynomial of the design point, and two gains can only come close to it. The schedule therefore chooses, at the measured speed, the pair whose coefficients of s^3 to s^0 come closest in

weighted least squares to those the tuned gains give at the design speed, as shown in Equation (9.12).

$$\min_{k_\varphi, k_{\dot\varphi}} \sum_{i=0}^{3} \left(\frac{a_i(v; k_\varphi, k_{\dot\varphi}) - a_i(v_d)}{a_i(v_d)} \right)^2 \tag{9.12}$$

From case 5's design, 40 N m/rad and 50 N m s/rad at 2 m/s, it gives 7.6 and 6.2 at 5 m/s and 4.2 and -2.9 at 8 m/s. Near 0.58 m/s, where the steer's grip passes through zero, the gains run away.

9.5.3 The Full-State Regulator

The regulator works on the declared bicycle with the motor's lag as a fifth state, $\dot{\mathbf{x}} = \mathbf{A}_5(v)\mathbf{x} + \mathbf{b}_5 T_{cmd}$ with $\mathbf{x} = [\varphi, \delta, \dot\varphi, \dot\delta, T_\delta]$, discretized for a zero-order hold at T_s by the framework's Riccati module, `shared/src/framework/core/riccati.js`, with Bryson's weights (Section 1.4.5) $\mathbf{Q} = \mathrm{diag}(1/\bar\varphi^2, 1/\bar\delta^2, 0, 0, 0)$ and $R = 1/\bar T^2$ on the lean, steer and torque scales $\bar\varphi$, $\bar\delta$ and $\bar T$. The law regulates about the steady turn that a lean reference asks for, as shown in Equation (9.13).

$$\begin{aligned} T_{cmd} &= T_r - \mathbf{K}(v)(\hat{\mathbf{x}} - \mathbf{x}_r), \qquad \mathbf{x}_r = [\varphi_r,\ \delta_r,\ 0,\ 0,\ T_r] \\ \delta_r &= -K_{\varphi\varphi}\varphi_r / K_{\varphi\delta}, \qquad T_r = K_{\delta\varphi}\varphi_r + K_{\delta\delta}\delta_r \end{aligned} \tag{9.13}$$

where K_{ij} are entries of $g\mathbf{K}_0 + v^2\mathbf{K}_2$ and δ_r is clamped to $\pm 20^\circ$ near the speed where $K_{\varphi\delta}$ vanishes. Fixed, $\mathbf{K}$ is designed at v_d, and at 5 m/s it is $[-159.1,\ 69.5,\ -26.4,\ 4.6,\ 0.39]$ in N m/rad for the angles, N m s/rad for the rates and N m per N m for the torque. Scheduled, a table of designs every 0.25 m/s from 0.25 to 12 m/s is computed once per model and weighting and interpolated linearly at the measured speed. At 1 m/s the scheduled gain on the lean rate is 203 N m s/rad, so the gyro's 0.1 °/s of noise reaches the command as 0.35 N m RMS through that gain alone, and as 0.44 N m RMS in all.

9.5.4 The Rider Model

The rider's balance reflex acts on the lean error and its rate $\tau_r = 150$ ms late, while the arms' own stiffness and damping on the bars act at once, as shown in Equation (9.14).

$$T_{cmd}(t) = k_r\left[(\hat\varphi - \varphi_r) + T_d\hat{\dot\varphi}\right]_{t-\tau_r} - k_a\hat\delta - c_a\hat{\dot\delta} \tag{9.14}$$

The delay is a buffer of the controller's own samples. A delayed reflex alone cannot ride this bicycle. With the arms' stiffness k_a set to zero, none of 36 pairs of (k_r, T_d) tried stays up even at 4 m/s, while with the arms the model rides at 3 and 4 m/s and falls at 2 and 1.5 m/s. Delayed reflexes and the passive impedance of the arms are both elements of the rider models the literature reviews [9, 16].

9.5.5 Path Loop

The path loop never touches the bars. A car follows a path by steering it directly, as the kinematic bicycle of Volume 20, Chapter 2 does, while a bicycle can only follow a path by leaning. The path loop turns the lateral error $e_y = y_{ref} - y_m$ and the heading error $e_\psi = \psi_{ref} - \psi_m$ into a lean reference, with the course's curvature κ read t_p ahead, where the bicycle will be once it has leaned, as shown in Equation (9.15).

$$\varphi_r = \frac{\omega_p^2 e_y + 2\zeta_p\omega_p v e_\psi}{g} + \arctan\frac{v^2\kappa(x + vt_p)}{g} \tag{9.15}$$

limited to $|\varphi_r| \le 15^\circ$, since a balanced bicycle accelerates sideways at about $g\varphi$. The course is a cosine through eight gates 15 m apart, alternately 1 m to the right and to the left, each 0.75 m wide on either side of its center, faded in and out by a quintic envelope so that its heading and curvature are continuous.

Table 9.5 The Nine Cases and the Challenge

#	Configuration and events	Lesson
1	manual at 1, 3 and 5 m/s, released at 2°	speed decides balance
2	hands off at 3, 5 and 8 m/s from 2°	the self-stable window
3	trail 0.12, 0, −0.02 m, tilt 10 and 5°, at 5 m/s	geometry moves the window
4	PD (40, 50) at 2 m/s, wrong sign	steer into the fall
5	same PD, speed 2 to 8 m/s at 0.5 m/s^2 from 3 s	one tuning, a schedule
6	scheduled LQR at 5 m/s, lean SP −5° at 2 s	countersteering
7	scheduled LQR and path loop, eight gates at 5 m/s	cascade order
8	rider at 4 m/s, shove at 5 s	reaction time
9	scheduled LQR at 0.3 m/s from 2°	honestly infeasible
C	slalom at 7, 3, 7 m/s, gust at gate 4, pothole at gate 6	everything at once

Table 9.6 Outcomes Read From the Running Engine

Case and condition	Result
1, hands off at 1 / 3 m/s	falls 1.46 / 2.00 s
1, a tap right at 5 m/s	leans 11.1° left, turns 38° left in 5 s
2, hands off at 5 / 8 m/s	under 0.5° by 5.5 s / falls 20.2 s
3, trail 0.12 / 0 / −0.02 m	window 4.70–7.11 / 2.93–3.13 m/s / none
3, tilt 10 / 5°; 20 kg passenger	4.68–5.45 m/s / none; 4.47–5.89 m/s
4, wrong sign / hands off / right sign	falls 0.61 / 1.66 s / within 0.5° from 0.93 s
5, fixed / scheduled	saturated 13.9 of 30 s / never, ≤1.58 N m
6, first steer; settled turn	4.3° right by 2.13 s; 2.1° left, 29 m
7, ω_p 0.6 / 2.5 rad/s; raw accelerometer	8 gates within 0.09 m / 4 missed; falls 7.19 s
8, rider at 4 / 2 / 1.5 m/s	lean peak 12.0° / falls 6.00 / 1.91 s
9, any of 123 designs	all fall; shipped LQR at 1.53 s
C, shipped / suggested	2 gates, fall 17.0 s / 8 gates, 0.10 % saturated

9.5.6 Manual Mode, Transfer and the Loop as It Runs

In manual mode the station's torque, a held arrow key (2 N m, ramped over 0.05 s) and a tap on either side of the drawn bicycle (2 N m for 0.2 s) push the bars. Every law keeps running, so on the switch to automatic the law's output starts from the hands' torque and the difference decays with a 0.1 s time constant. The saturation flag compares the law's request with $\pm T_{max}$ and is cleared every period. The tilt switch cuts the motor in either mode.

The Map view, the balance card and the export report the closed loop as it runs. The bicycle with the motor's lag is held over each period, the law is applied once a period, and each discrete root z is shown as $\ln z/T_s$. For the rider, a second-order Padé approximant of the delay $\tau_r + T_s/2$ stands in for the buffer. The hold decides stability for the lean PD, whose lean-rate gain drives a closed-loop oscillation of roughly 9 to 25 rad/s between 2 and 8 m/s. Applied continuously, the fixed PD of case 5 would be stable up to 5.59 m/s, and as it runs it is unstable from 3.45 m/s, as shown in Figure 9.2b. Its schedule, which the continuous analysis rates stable on 2 to 8 m/s with the weakest root at –0.062 s^{-1} near 3.6 m/s, lets the closed loop's oscillation grow at up to 0.157 s^{-1} near 3.47 m/s as it runs. The engine is the referee. With the true state, the noise off and the speed held, the lean's envelope under the schedule grows at 0.082, 0.142 and 0.086 s^{-1} at 3, 3.5 and 4 m/s, against 0.082, 0.156 and 0.086 from the sampled analysis and −0.190, −0.066 and −0.095 from the continuous one. The scheduled LQR's largest real part stays below –2.76 s^{-1} from 0.5 to 10 m/s.

9.6 Nine Cases and a Slalom, Measured

The cases are listed in Table 9.5 and their outcomes in Table 9.6. Every number in this section was measured on the engine run headless with the page's own stepping, at the default seed, with any change a sentence names entered before Run, and every time is measured from Run. A case opened first, one chosen from the list after another case and one restarted with Reset give identical runs.

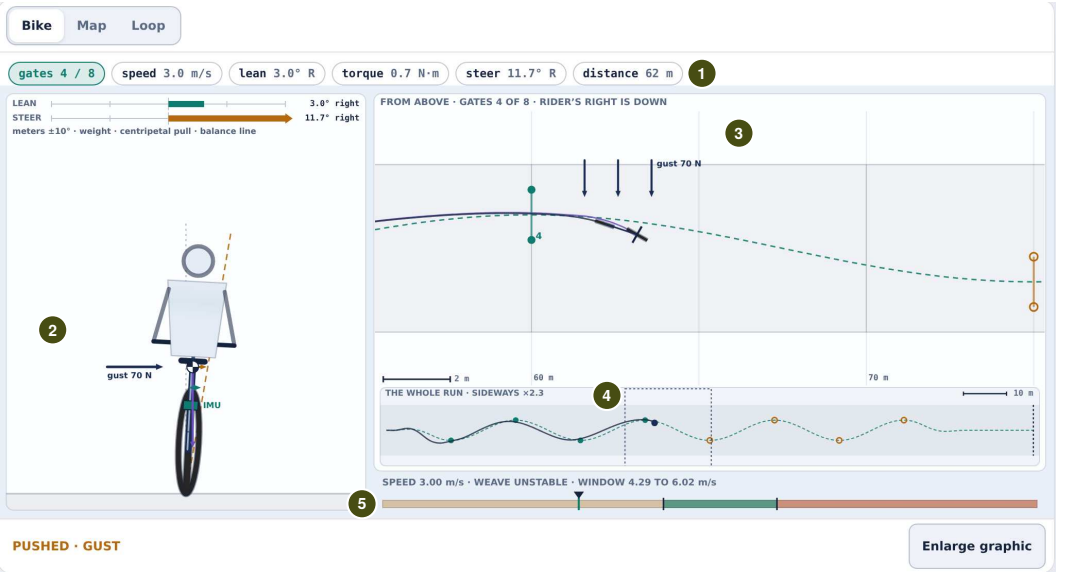


Figure 9.3 The slalom on its speed program with the suggested scheduled LQR, captured at $t = 11.4$ s on the Bike view and cropped to the plant view (the header, the control panel, the trends and the scorecard are cut). (1) The badges read 4 of 8 gates, 3.0 m/s, a lean of 3.0° to the right, a torque of 0.7 N m, a steer of 11.7° to the right and 62 m. (2) In the rear view the 70 N gust at gate 4 pushes from the left at the center of mass, and the meters show the bicycle leaning 3.0° to the right while the regulator steers 11.7° to the right into the fall. (3) From above, the three gust arrows cross the road just past gate 4. (4) The whole run, stretched sideways, with the window of the view above it. (5) The speed bar puts 3.00 m/s in the weave band below the window of 4.29 to 6.02 m/s, and its label reads that the weave is unstable.

9.6.1 By Hand, and Hands Off

Case 1 releases the bicycle at 1 m/s with the student's hands on the bars. A held key or a tap pushes with 2 N m, and pushing into the fall, to the right, brings the wheels back under the bicycle, but its fastest motion doubles every 0.20 s and hands off it is down in 1.46 s. At 3 m/s a player who steers by the lean they see, only 25 ms late, is down within 4.5 s for the best of 49 gain pairs on the keys' 2 N m. At 5 m/s the student can let go, and one tap to the right leans the bicycle 11.1° to the left and turns it 38° to the left in 5 s, a first meeting with case 6. Case 2 runs the same push with nobody steering and measures the window of Section 9.4. Case 3 moves it. The window is 4.70 to 7.11 m/s at 0.12 m of trail and 2.93 to 3.13 m/s at zero trail, so trail is one term among several. At −0.02 m there is no window, and at 5 m/s a real mode growing at 0.65 s^{-1} brings the bicycle down at 4.37 s. A steer axis tilt of 10° gives 4.68 to 5.45 m/s and 5° gives none, and a 20 kg passenger, which the controller does not know about, narrows the window to 4.47 to 5.89 m/s.

DID YOU KNOW? No Trail, Still a Window

Experiments of 1970 pointed to trail as an important factor in a bicycle's stability [1]. From the simulator: with no trail at all, the benchmark bicycle still rides itself between 2.93 and 3.13 m/s, while a steer axis tilted only 5° leaves no window at any speed (Table 9.6).

9.6.2 The Sign, and One Tuning Across Speeds

Case 4 balances at 2 m/s with the lean PD at 40 N m/rad and 50 N m s/rad, shipped with the wrong sign. It steers away from the lean and throws the bicycle down in 0.61 s, against 1.66 s with nobody steering, its largest closed-loop root at +13.7 s^{-1} against the bicycle's own +2.68 s^{-1}. With the right sign the slowest mode decays at 0.86 s^{-1} and the lean is within 0.5° of upright from 0.93 s with at most 2.78 N m.

Case 5 carries the same PD up a ramp from 2 to 8 m/s at 0.5 m/s^2 from 3 s. The torque first saturates at 11.9 s, at 6.44 m/s, and is saturated for 13.9 of the 30 s. From 20 s the bars swing ±7.7° at 2.9 Hz while the lean moves only ±1.8°, a saturated limit cycle (Section 1.4.9) of the closed loop's single oscillatory pair, which the linear analysis puts at 3.91 Hz at 8 m/s, well above the bicycle's own weave at 1.35 Hz. Scheduled, the same design rides the whole ramp with at most 1.58 N m and never saturates. The weak band of Figure 9.2c does not show, because the ramp crosses it in 3.0 s, in which a growth of at most 0.157 s^{-1} enlarges a disturbance by at most a factor of 1.6. In the band the growing pair oscillates at 1.50 Hz at 3.5 m/s with the steer swinging about five and a half times the lean, the weave as the feedback has moved it, against 0.42 Hz for the bicycle alone. Stabilizing gain pairs exist at every speed, but choosing them would be a new design rather than one design carried across speeds. The full-state LQR, which also feeds back the steer, has no such band, and the PD schedule's band is kept as a teaching result.

9.6.3 Countersteering and the Slalom

Case 6 holds the bicycle upright at 5 m/s with the scheduled LQR and asks for a 5° lean to the left at 2 s. The torque goes right first, up to 7.1 N m, the steer swings 4.3° to the right by 2.13 s and turns left only from 2.25 s, the lean reaches −4.5° at 2.42 s, and the turn settles at 2.1° of left steer on a circle of about 29 m. With the true state in the loop the radius is within 0.4 % of $v^2/(g\tan\varphi)$. The rear wheel's track moves a few centimeters to the right first, 2.6 cm in this run and 5.9 cm in the noise-free linear model.

TRY IT IN THE APP 6 · Countersteer: How a Left Turn Starts

Set the clock to 0.25×, press Run and watch the steer angle trend when the left turn is asked at 2 s. The bars swing 4.3° to the right first, by 2.13 s, and turn left only from 2.25 s. Only then does the lean follow, reaching −4.5° at 2.42 s.

Case 7 rides the eight gates at 5 m/s under the path loop and the scheduled LQR. At $\omega_p = 0.6\,\mathrm{rad/s}$ every gate is passed within 0.09 m of its center, the loop asking for at most 8.6° of lean. At 2.5 rad/s it asks for leans up to its 15° limit, the lean loop delivers each one late and after a countersteer, and the bicycle misses 4 of the 8 gates by up to 1.21 m. Linearized with the true state, that cascade has a root at +3.13 s^{-1} at 5 m/s, against −0.54 s^{-1} at the shipped 0.6 rad/s, and only the lean and torque limits hold it to a wide swing. With the raw accelerometer the slalom's turns pull the lean estimate 27° from the truth and the bicycle falls at 7.19 s.

9.6.4 A Rider, and a Walking Pace

Case 8 hands the bars to the rider model at 4 m/s. It catches a shove at the saddle with the lean peaking at 12.0° and at most 2.03 N m on the bars. At 1.5 m/s it falls at 1.91 s, and at 2 m/s at 6.00 s. At 1.5 m/s the bicycle's weave grows at 3.11 s^{-1}, doubling every 0.22 s, and 150 ms of delay is most of that time, the delay limit of Section 1.4.7. The LQR has no delay, but braking to a stop at 0.5 m/s^2

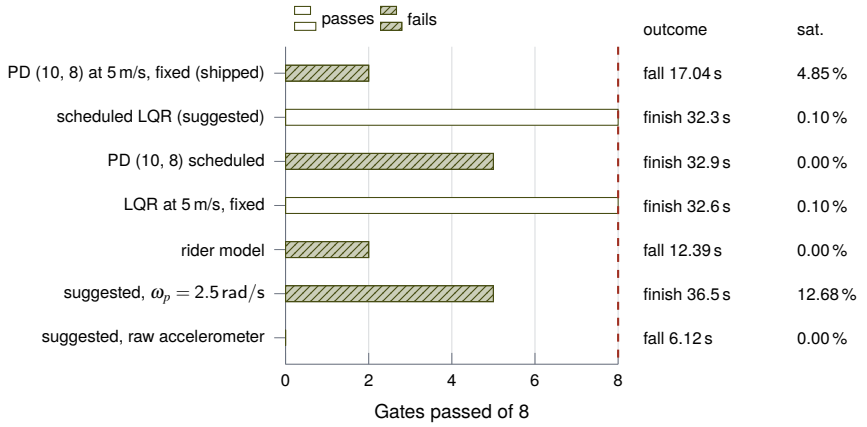


Figure 9.4 The slalom configurations of Table 9.7: gates passed of eight (the target, dashed line), with each run's outcome and the share of the 40 s with the torque command saturated. White bars pass the challenge, hatched bars fail it. Notice that both passing runs carry a regulator, while the scheduled PD finishes the course with only five gates, and the fast path loop finishes with five gates and 12.68 % of the run saturated.

it holds on only until the bicycle has all but stopped. In the last few centimeters per second it saturates the motor and puts the bars on their stop, below 0.17 m/s and at 0.05 m/s in this run, and it falls as it stops. Its gain on the lean grows from 194 N m/rad at 3 m/s to 1650 N m/rad at 0.5 m/s, as the grip of Section 9.4 predicts.

Case 9 asks the scheduled LQR to hold the bicycle at 0.3 m/s after the same 2° push. The design is stable on paper, its closed loop decaying at 2.53 s^{-1}, but from a perfect reading of the lean it asks at once for 123 N m, about twelve times what the motor has. The bars reach their stop at 0.21 s and the bicycle falls at 1.53 s. The application's own sweep from this start runs 123 designs, 80 LQR weightings, 40 PD gain pairs of either sign, the PD schedule, the rider and the LQR given the true state, and every one falls. A smaller push is another matter. From 0.25° the LQR holds the bicycle with the steer swinging to 24.8°, and the 2° push is held at 1 m/s with 20.9° of steer and at 2 m/s with 7.8°. At a walking pace riders put a foot down.

9.6.5 The Slalom on a Speed Program

The challenge rides the course of case 7 with the speed setpoint at 7 m/s, 3 m/s from 40 m and 7 m/s again from 100 m, through a 70 N gust at gate 4 and the pothole at gate 6. Its target is no fall, all eight gates, the finish line at 150 m, and the torque command saturated for at most 2 % of the 40 s, judged on the truth, with the score the count of gates. The configurations are compared in Table 9.7. As shipped, a lean PD tuned at 5 m/s, where the bicycle rides itself, passes 2 gates and falls at 17.0 s at 3.00 m/s, 11 m after the gust. The suggested scheduled LQR passes all eight within 0.40 m of their centers and finishes at 32.3 s. At seeds 1 to 5 the shipped settings fall between 14.75 and 17.04 s with 2 gates and the suggested ones pass with their worst gate 0.39 to 0.44 m off center. The state of the suggested run at 11.4 s, inside the gust and the weave band, is shown in Figure 9.3, and Figure 9.4 draws the gates and the saturation of every configuration.

9.7 Technology in Practice

What Measures It

A riderless test bicycle built to check a model of the benchmark's kind measured the roll rate, the yaw rate, the steer angle and the rotation of the rear wheel, and its eigenvalues were fitted to the measured lean rate [11]. A balance-assist bicycle reads its roll rate with a micromachined rate gyroscope and

Table 9.7 The Slalom Over 40 s, Default Seed

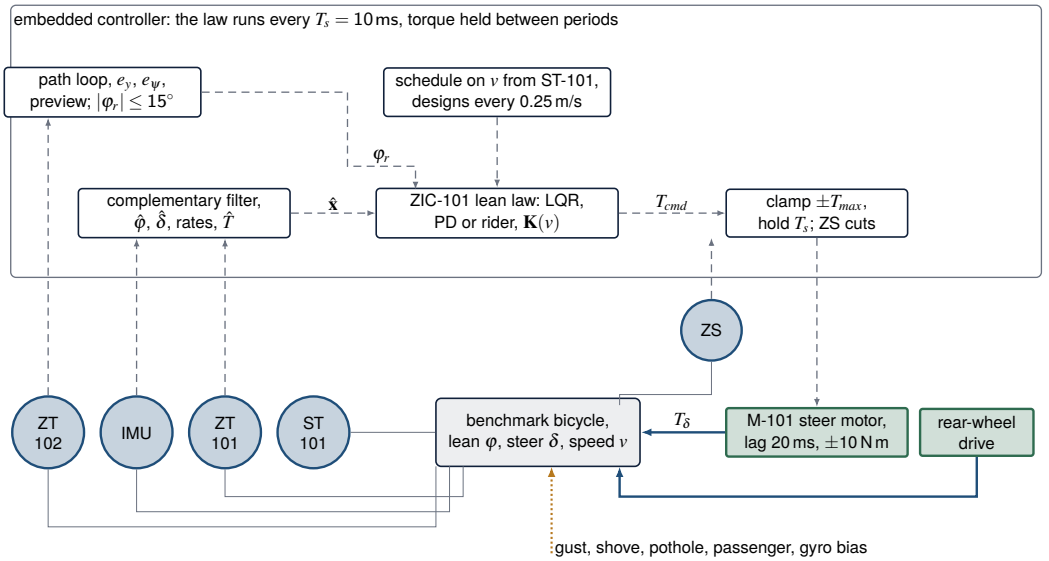
Configuration	Gates	Outcome	Sat.	Verdict
PD (10, 8) at 5 m/s, fixed (shipped)	2	fall 17.04 s	4.85 %	fails
scheduled LQR (suggested)	8	finish 32.3 s	0.10 %	passes
PD (10, 8) scheduled	5	finish 32.9 s	0.00 %	fails
LQR at 5 m/s, fixed	8	finish 32.6 s	0.10 %	passes
rider model	2	fall 12.39 s	0.00 %	fails
suggested, $\omega_p = 2.5\,\mathrm{rad/s}$	5	finish 36.5 s	12.68 %	fails
suggested, raw accelerometer	0	fall 6.12 s	0.00 %	fails

the speed of its rear wheel with a wheel speed sensor [17]. The chapter's instruments have the same shape: a gyro with noise and bias, an accelerometer that cannot tell gravity from the turn, an encoder on the steer and an exact wheel speed, merged by the complementary filter of Equation (9.10).

> **TECHNOLOGY** **Sensor: Rate Gyros on the Frame, an Angle Sensor on the Steer**
>
> One riderless test bicycle carried two angular rate gyros, rated to $\pm 100\,°/\mathrm{s}$, clamped at right angles on the seat post to read the lean rate and the yaw rate, with a spirit level to set them square. A potentiometer coupled straight to the steering axis read the steer angle, and a ten-pole magnet ring on the rear wheel gave the speed from the time between zero crossings, a staircase on the trend. Sampling was raised from 100 to 400 Hz to keep the speed signal from aliasing, but the steer signal then turned erratic [11]. The chapter's encoder has 2^{14} counts a turn, and its wheel speed is exact.

What Moves It

Real machines use two kinds of steering actuator. The balance-assist bicycle applies a torque between the head tube and the steer tube, and the motor's torque and the rider's act on the steer in sum [17]. A bicycle robot built for vehicle testing treats its steer angle as perfectly controlled, so its model runs from the steer angle to the lean and leaves the steer torque out [18]. The chapter's steer motor is of the first kind, a torque with a 20 ms lag and a limit of 10 N m, which is why case 9 can ask for more torque than the motor has.



IMU: rate gyro and accelerometer at h_s; ZT-101 steer encoder, 2^{14} counts a turn; ST-101 wheel speed; ZT-102 lane position

Figure 9.5 The balance loop on test-bicycle hardware. The inertial unit on the frame (rate gyro and accelerometer) and the steer encoder ZT-101 feed the complementary filter, the lane position ZT-102 feeds the path loop, which commands the lean, and the wheel speed ST-101 picks the gains from the schedule. The lean law ZIC-101 sends a torque command, clamped and held for one period, to the steer motor M-101, and the tilt switch ZS cuts the motor at a fall. Notice that speed enters the loop twice: it moves the bicycle's own poles and it chooses the controller's gains.

> **TECHNOLOGY** **Final Element: A Steering Motor in Parallel With the Rider**
>
> A balance-assist electric bicycle carries a motor in the steering assembly that applies up to 7 N m between the head tube and the steer tube. A motor controller turns the commanded torque into applied torque, and the rider's torque on the bars adds to it. The law is roll rate feedback with a gain that falls linearly as the speed rises toward about 4.7 m/s. With the higher gain it stabilizes the riderless bicycle's weave down to 3.4 km/h [17]. The chapter's motor is a torque source with a 20 ms lag and a limit of 10 N m, and the dummy's hands put no torque on the bars.

Where the Controller Runs

On the balance-assist bicycle the law runs on a small microprocessor carried on the bicycle, and every sensor is logged on a vehicle data bus at 100 Hz or faster [17]. An earlier test bicycle recorded through a data acquisition unit with 14-bit inputs into a laptop fixed on its rear rack, where the laptop's mass reduced the bicycle's stability and was weighed as part of the measured machine [11]. The chapter's controller runs every 10 ms on integration steps of 1 ms and holds its torque between periods, and that hold alone moves the stability edge of the lean PD tuned at 2 m/s from 5.59 to 3.45 m/s. The block that holds the chapter's structure is the schedule, a table of regulator designs interpolated at the measured speed. Figure 9.5 draws the loop on test-bicycle hardware.

What Is Modernizing It

The first development is balance assistance on consumer electric bicycles. One study added a steering motor to such a bicycle and, with 26 participants riding on a treadmill at 6 km/h and pushed at the handlebars, found the chance of recovering significantly higher with the assistance on [17]. The second is the self-balancing bicycle robot as a cyclist target for testing driver assistance systems. Unlike a stable moving platform, such a robot is unstable and so leans more naturally [18]. The third

is the schedule built from data. Gray-box identification at a few speeds replaces measuring every part, and the next steps named are adapting the model to changing loads, such as different dummies, by adaptive control and online identification [18]. The chapter's regulator trusts the declared bicycle, and its 20 kg passenger shows what such a learner would have to correct.

IN PRACTICE Commissioning a Schedule From Closed-Loop Tests

A bicycle robot was tuned without taking the bicycle apart. Its balance loop was closed first, and periodic multisine signals up to 25 Hz were added at 2.2, 2.4, 2.6 and 2.8 m/s. Averaging over the periods removes the bias that closing the loop puts into a frequency response. A gray-box model with the speed as a parameter was fitted to all four, and the gains were scheduled for the same bandwidth at every speed: 1.5 Hz for the inner balance loop and 0.15 Hz for the outer lean loop. The schedule then held the robot at 1.7 m/s, outside the tested range [18]. The chapter's path loop is likewise kept far slower than its lean loop.

9.8 Limitations

The bicycle is the Whipple model linearized about upright running, so it has no tires, no frame flex, no wobble and no rider moving on it, and its lean is trusted only for small angles. The matrices follow the speed quasi-steadily, which the benchmark does not cover, the nonlinear option is gravity alone and ends the capsize in a steady turn rather than a fall, and the cross slope's lean torque is a stand-in that knife-edge wheels would not feel. The motor, the stop, the fall angle, the instruments and every threshold are teaching values rather than any test bicycle's.

The Map and Loop views are linear analyses on the true state, with the motor's lag and the controller's sampling but without noise or the estimator, and the rider's delay enters them through a Padé approximant. The two-gain PD schedule leaves a band near the weave speed in which the closed loop grows slowly, which the case teaches rather than repairs. The regulator and the schedules trust the declared bicycle, and no estimator learns a passenger or a gyro bias. The rider model is a control law with passive arms, not a fitted model of a person.

OPEN CHALLENGE How a Stabilized Bicycle Feels to Its Rider

A steering motor that stabilizes a bicycle also changes the machine the rider must steer. On one balance-assist bicycle the controller raised the weave frequency about threefold, to almost 1 Hz, which a rider can still follow but which may feel like a normal bicycle at 25 km/h. Riders of a similar bicycle reported dissatisfaction. Fewer falls in a laboratory cannot yet be turned into fewer falls on the road, because detailed data on how people fall in natural riding are missing. A long study of normal use against a control group would give the strongest evidence [17]. The rider of this chapter is a control law, not a fitted model of a person.

9.9 Conclusion

The simulator of this chapter derives the benchmark bicycle's four matrices from its geometry and masses, to 1.4×10^{-14} of the published ones, and its eigenvalues reproduce the published table to 6.2×10^{-13}. The same derivation runs the bicycle. Hands off, it falls in 2.00 s at 3 m/s, rides itself at 5 m/s and tips over in 20.2 s at 8 m/s, and a trail of −0.02 m removes the window. A lean loop with the wrong sign falls in 0.61 s against 1.66 s with nobody steering, and gains tuned at 2 m/s are unstable above 3.45 m/s once the controller's 10 ms hold is counted. Their two-gain schedule rides a ramp to 8 m/s without saturating but leaves a band from 2.81 to 4.29 m/s where the closed loop grows at up to $0.157\ \mathrm{s}^{-1}$, and the scheduled regulator holds every speed from 0.5 to 10 m/s. A left turn starts with 4.3° of right steer, a path loop must stay slower than the lean it commands, and at 0.3 m/s, where the steer's grip on the lean has reversed, none of 123 designs holds the bicycle up. The bicycle is linear, rigid and riderless. A fuller model would add tire relaxation and frame compliance for wobble, a nonlinear model that falls as the real machine does, a rider who moves on the frame and an estimator for the gyro's bias. The simulator is a bicycle whose speed decides which controller it needs.

KEY POINTS

- A bicycle with nobody on it rides itself only between two speeds, 4.292 and 6.024 m/s.
- The steer's grip on the lean grows with the square of the speed, passes through zero at 0.58 m/s and is reversed below.
- A left turn starts with 4.3° of right steer, and a path loop must stay slower than the lean it commands.
- Gains tuned at 2 m/s are unstable above 3.45 m/s once the 10 ms hold is counted, while the scheduled regulator holds every speed from 0.5 to 10 m/s.

Where else this applies. Chapter 4 meets a gain that changes sign with the operating point, and Volume 3, Chapter 10 a wrong-way start.

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9.10 Exercises

9.1 Concept check · 2 min

In case 2 the bicycle is released hands off at 8 m/s from a 2° lean. It leans slowly into a tightening spiral and passes 40° after 20.2 s. Which mode brings it down?

(a) The weave, whose real part turns positive again above the capsize speed and grows slowly

(b) The castering mode, which at 8 m/s becomes too fast for the steer to follow without control

(c) A limit cycle of lean and steer that starts when the steer reaches its 45° stop

(d) The capsize, a slow real mode dominated by lean that grows at 0.143 s^{-1} at this speed

9.2 Concept check · 1 min

In case 6 the scheduled LQR holds the bicycle upright at 5 m/s, and at 2 s it is asked for a 5° lean to the left. Which way do the bars turn first, and why?

(a) Right, by about 4.3°, so that the wheels run out from under the bicycle and it starts to lean left

(b) Left, by about 4.3°, because the regulator steers toward the turn and the lean then follows the steer

(c) Right, because the spinning front wheel's gyroscopic torque reverses the steer above the weave speed

(d) Neither way: the regulator holds the steer at zero until the lean has reached the 5° requested

9.3 Concept check · 2 min

In case 9 the scheduled LQR, stable on paper at 0.3 m/s, lets the bicycle fall from a 2° push, and so do 122 other designs. What is the reason?

(a) The LQR's gain table has no design at so low a speed, so the regulator runs with the gains of a faster bicycle

(b) The steer's grip on the lean has reversed and shrunk, so catching the push needs far more torque and steer than the motor and the stop allow

(c) The gyro's noise, multiplied by the very large scheduled gains, keeps the motor saturated before the push can be caught

(d) The weave grows at 3.53 s^{-1} at this speed, which is faster than a controller sampling every 10 ms can follow

9.4 Theory and calculation · 6 min

Use the printed matrices of Equation (9.5) with $g = 9.81\,\mathrm{m/s^2}$, $w = 1.02\,\mathrm{m}$ and $\lambda = 18°$.

(a) Find the gravity part of the steer's grip on the lean, $gK_{0\varphi\delta}$, the grip $gK_{0\varphi\delta} + v^2K_{2\varphi\delta}$ at 5 and 1 m/s, and the speed at which the grip is zero. *Checkpoint:* −25.5 N m/rad, 1889 N m/rad, 51.1 N m/rad and 0.577 m/s.

(b) In a steady turn the rates are zero and $T_\varphi = 0$, so the lean row of Equation (9.1) fixes the steer that holds a lean. Find δ/φ at 5 m/s, and the steer for a 5° lean to the left. *Checkpoint:* 0.420, so 2.10° to the left.

(c) Find the radius of that turn from Equation (9.6), and compare it with $v^2/(g\tan\varphi)$. *Checkpoint:* 29.2 m against 29.1 m.

9.5 Theory and calculation · 6 min

Use the eigenvalues of Table 9.3.

(a) At 1 m/s, find the doubling time of the weave and its period. Explain why the fall at 1 m/s does not look like a weave. *Checkpoint:* 0.197 s and 7.8 s.

(b) At 8 m/s, find the e-folding time of the capsize, and the time a pure exponential would take to grow from 2° to 40°. Compare with the 20.2 s measured. *Checkpoint:* 6.98 s and 20.9 s.

(c) In case 5 the drive ramps the speed at 0.5 m/s^2. How long does the bicycle spend in the band from 2.81 to 4.29 m/s, and by what factor at most can a growth of 0.157 s^{-1} enlarge a disturbance there? *Checkpoint:* 2.96 s and 1.59.

9.6 Theory and calculation · 6 min

The lean estimate of Equation (9.10) runs every $T_s = 10\,\text{ms}$ with $\tau_{cf} = 0.5\,\text{s}$.

(a) Find the filter coefficient a, and the share of each new accelerometer reading that enters the estimate. *Checkpoint:* 0.980 and 2.0 %.

(b) At 1 m/s the scheduled LQR's gain on the lean rate is 203 N m s/rad. Find the RMS torque that the gyro's 0.1 °/s of noise puts on the command through that gain. *Checkpoint:* 0.35 N m.

(c) The bicycle runs a steady left turn at 5 m/s with 5° of lean and 2.10° of steer. Neglect noise and $h_s\ddot{\varphi}$, and find what the accelerometer reads, $\varphi - v\dot{\psi}/g$ with $\dot{\psi}$ from Equation (9.6). *Checkpoint:* about $-0.01°$, almost no lean.

9.7 App challenge · 8 min

Open case 4, "Steer into the fall: the lean loop's sign", at the default seed and 1× speed.

(a) Press Run as shipped and let it fall. Read the Fall tile, and say which way the bars turn relative to the lean. *Checkpoint:* 0.61 s, away from the lean.

(b) Switch to Manual and press Hands off, so the steer torque reads 0 N m (until then the hands keep the loop's last torque). Press Reset and Run, and read the Fall tile. *Checkpoint:* 1.66 s.

(c) Switch back to Auto, set Feedback sign to Into the fall, press Reset and run for 10 s. Read the Largest torque tile, and from the lean trend or the CSV column `truth_lean_deg` the time after which the lean stays within 0.5° of upright (to 0.05 s). *Checkpoint:* 2.8 N m and about 0.93 s.

9.8 App challenge · 8 min

Open case 5, "One tuning from 2 to 8 m/s", at the default seed and 2× speed. The PD tuned at 2 m/s rides a speed ramp to 8 m/s that starts at 3 s.

(a) Run the full 30 s as shipped and read the Steer saturated tile. From the CSV columns `steer_saturated` and `speed_mps`, find the time and speed of the first saturation. *Checkpoint:* 13.94 s (46.5 %); first at 11.9 s, at 6.44 m/s.

(b) From `truth_steer_deg` and `truth_lean_deg` between 20 and 30 s, find the swing of the steer and of the lean and the frequency of the steer. Is this the bicycle's own weave, at 1.35 Hz at 8 m/s? *Checkpoint:* ±7.7°, ±1.8°, about 2.9 Hz.

(c) Press Reset, set Gain schedule on speed to On, and run the 30 s again. Read the Steer saturated and Largest torque tiles. *Checkpoint:* 0.00 s and 1.6 N m.

9.9 App challenge · 8 min

Open the challenge, "The slalom on a speed program", at the default seed and 5× speed. Change nothing but the controller settings named, each before pressing Run, and let every run go the full 40 s.

(a) Run it as shipped (lean PD tuned at 5 m/s). Read the Gates, Fall and Steer saturated tiles. *Checkpoint:* 2 of 8, at 17.04 s, 4.9 %.

(b) Press Reset, set Controller to Full-state LQR and Gain schedule on speed to On, and run again. Read the same tiles and the Challenge target. *Checkpoint:* 8 of 8, none, 0.1 %, met.

(c) Press Reset and set Gain schedule on speed to Off, so the LQR is designed at 5 m/s only. Run again. Then press Reset, set Controller to Lean PD with the schedule On, and run once more. Which passes? *Checkpoint:* fixed LQR 8 of 8 and met; scheduled PD 5 of 8, not met.

9.10 Application area · 3 min

The simulated bicycle is the Whipple model with knife-edge wheels.

(a) Real bicycles and motorcycles can show wobble, a fast steering oscillation. Why does the simulator never show it, and what would a model need to show it? *Checkpoint:* no tire or frame compliance; add tire relaxation and frame compliance.

(b) The road's cross slope enters Equation (9.8) as a lean torque. Can the simulator be used to judge how a real crowned road affects a test bicycle? *Checkpoint:* no; the term is a teaching stand-in.

9.11 Application area · 4 min

A delivery bicycle with a steer assist carries a 20 kg child seat on its rear rack. The assist's model is the empty bicycle, as in the simulator.

(a) The passenger narrows the self-stable window from 4.29–6.02 m/s to 4.47–5.89 m/s. Which matrices change, and which does the controller use? *Checkpoint:* all four the physics integrates; the declared, empty ones.

(b) How would you check, with the simulator, that the scheduled LQR still serves the loaded bicycle across its speed range, and what would you change in the product if it did not? *Checkpoint:* passenger slider, Map view and runs across speed; schedule on a measured load or design for the range.

9.12 Review the agent · 5 min

An AI assistant reviews the lean PD of case 5 for a delivery bicycle that cruises at 3.5 m/s. It writes: "Scheduled from the 2 m/s design, the closed loop is stable from 2 to 8 m/s, with its weakest root at $-0.062\ \mathrm{s}^{-1}$ near 3.6 m/s. The schedule is safe to ship." Find the flaw, and say how you would check the claim on the simulator.