INTERACTIVE CONTROL ENGINEERING
Principles and Applications

Volume 7 Nonlinear Dynamics, Oscillations, and Stabilization

CHAPTER 10

Holding a Drifting Car at an Unstable Equilibrium With Steer and Throttle

Pages 241–262

Edited by John D. Hedengren

How to cite this chapter. "Holding a Drifting Car at an Unstable Equilibrium With Steer and Throttle," in *Nonlinear Dynamics, Oscillations, and Stabilization*, John D. Hedengren, Ed., vol. 7 of *Interactive Control Engineering: Principles and Applications*. Advanced Process Solutions, LLC, 2026, ch. 10, pp. 241–262.

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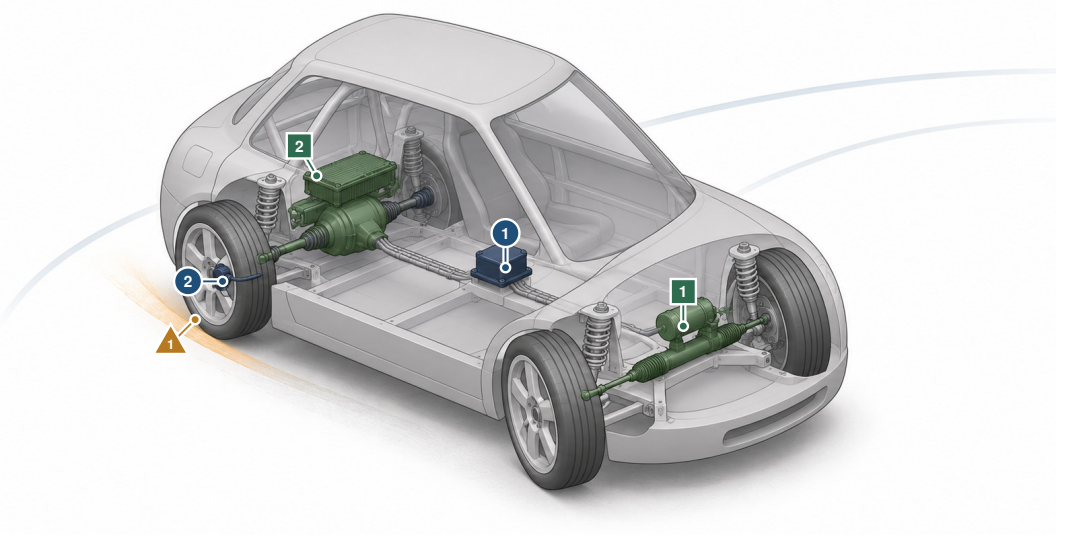
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CHAPTER 10

Holding a Drifting Car at an Unstable Equilibrium With Steer and Throttle



The controller moves 1 the steering servo on the front rack, which turns the front wheels
2 the electric drive on the rear axle, which sets the drive torque

The controller reads 1 the navigation unit with its gyro: sideslip, yaw rate, speed and position
2 the encoder that reads how fast the driven wheels spin

What disturbs it 1 the grip of the pad under the tires, lower on the wet patch

Notice that the rear tires slide at their grip limit, so the drive torque on the rear axle is the handle left on how hard they push sideways. Illustration, not to scale.

IN PLAIN TERMS **Holding a Car Sideways in a Slide**

The main idea. A drifting car circles a cone sideways, its rear tires sliding at the edge of their grip. A computer can hold it there by turning the front wheels and setting how hard the rear wheels are driven.

The challenge. Like a pencil balanced on its tip, the car left alone spins out within seconds. Its sliding rear tires cannot push sideways any harder, so the only handle left on them is their forward push, which takes sideways grip away as it grows.

Why it matters. Many machines must be held at a point they would leave on their own.

THREE WAYS TO READ THIS CHAPTER

Explore	11 min	the first page, the fact in the Introduction (Section 10.1), the history segment in Section 10.2, the try-its and the fact in Section 10.6, and the key points
Study	45 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	15 min	the first page, the results (Section 10.6), Technology in Practice (Section 10.7), Limitations (Section 10.8) and the open challenge

The control question A rear-drive car can circle a cone sideways in steady state, its rear tires sliding at the edge of their friction. That drift is an unstable equilibrium whose sliding tire cannot raise its lateral force. What handle is left, and which controllers can hold the car there?

Learning objectives

- Solve the force balances of a steady drift for the rear forces and the drive force on the friction circle.
- Read the two unstable roots of the drift from its linearization, and relate the slide's root to the growth measured on a trend.
- Explain why a yaw-rate grip controller spins the car and why two single loops on throttle and steer fight when the speed target moves.
- Compare a two-input LQR with the nested loops after a kick, and schedule its equilibrium and gain on a friction estimate.
- Show that a front-drive car has no saturated-rear drift when both axles see the same friction.

Prerequisites Sections 1.4.2 (steady states), 1.4.3 (gain schedule), 1.4.4 (eigenvalues), 1.4.5 (discrete LQR, Bryson's rule), 1.4.7 (limits from an unstable pole) and 1.6.1 (engine). Chapter 9 for an LQR scheduled on speed.

The app https://apmonitor.com/apps/autonomous-drift/

See also Volume 20, Chapter 2 for a steering loop over a proportional-integral speed loop on a kinematic bicycle, whose wheels roll without slipping.

10.1 Introduction

On a dry skidpad the simulated coupe can travel at 8 m/s with its nose 35° inside its direction of travel, yawing at 0.927 rad/s around a circle of 8.63 m radius. Its rear tire slips at −42.4° and pushes with all of the 5.12 kN its load and friction allow, its front wheels are turned 25.04° toward the outside of the turn, and the motor holds 3.72 kN of drive force at the rear contact. Every acceleration of the body is zero to 10^{-14}, so this drift is an equilibrium of the vehicle model. Started on it with the steering and the throttle held, the car is 5° off at 2.98 s and spun at 4.29 s.

Operating points that are useful only because feedback holds them are common. Statically unstable aircraft, balancing robots, magnetic bearings and an exothermic reactor at its middle steady state (Chapter 2) all ask a controller to hold a state the plant leaves on its own, and an unstable pole sets limits on every such loop that no tuning removes, among them a lower bound on the loop's bandwidth and a danger from actuator saturation [1] (Section 1.4.7). A car past the limit of its rear tires belongs to the same class. A bifurcation analysis of the nonlinear single-track model shows the stable cornering equilibrium losing its stability as the tires saturate [2], and the large-sideslip "powerslide" is a steady state whose stability can be analyzed in the same terms [3]. The drift adds a feature that the textbook unstable plants lack. The actuator that would normally correct the error, the rear tire's lateral force, is the one that is saturated, so the correction has to come through the coupling between two other inputs.

Two beliefs are the simulator's targets. The first is that a sliding tire has lost its authority. In the drift the rear lateral force follows the drive force along the friction circle, so the throttle still moves it, and in case 3 the error grows at 1.98/s, close to the linearization's 1.95/s, precisely because nothing in the open loop uses that handle. The second is that a controller that steers a car well on grip will hold it at the limit. The yaw-rate and speed loop of case 2, the structure of a road car's stability control, spins the car at 3.61 s when asked for 30° of sideslip and spins it for every request of 10° or more. A transfer function about a grip operating point shows neither, because the first is a property of the saturated equilibrium and the second a consequence of designing around the wrong one.

✦ DID YOU KNOW? What Stability Control Was Expected to Save, 2007

When it made stability control mandatory in 2007, the United States highway safety agency estimated that the systems would save 5,300 to 9,600 lives and prevent 156,000 to 238,000 injuries a year once every light vehicle on the road had one. It expected single-vehicle crashes of passenger cars to fall by 34 percent [4].

The work that established drifting as a control problem was done on instrumented research cars, and a student meets its pieces apart. Equilibrium analysis and phase portraits are given for fixed speed [5], state feedback is designed at one operating point [6], and the experimental controllers follow a path with the drift inside [7, 8]. This chapter puts the pieces together on one declared car run in a browser. It asks what it takes to hold that unstable equilibrium, from bare hands through two interacting single loops to a scheduled two-input LQR, and where the equilibrium stops existing. The single-track model, the brush tire, the proportional-integral loops and the LQR are all established methods. The results are measurements on one parameter set and seed. With the speed free the drift has two unstable roots (1.95/s for the slide and 0.21/s for the speed under a held torque), where the fixed-speed analysis finds one, 1.89/s. A short proof shows that the same car with front drive has no saturated-rear equilibrium when both axles see the same friction, and each figure-eight reversal takes the rear tire off its friction circle for an instant. The simulator uses one pure rate function alike in the integrator, the equilibrium solver and the linearization, a shared tire module with the brush model and its combined-slip form, a two-input LQR computed through the series' Riccati module, and a measurement boundary that keeps every law away from the simulation's truth. Ten cases run from a car driven by hand to a scored minute of donut and figure-eight, and the app credits each case only when its own rule is met.

Section 10.2 traces the domain from the handling limit to automated drifting. Section 10.3 gives the car, its tires and its drive. Section 10.4 solves for the drift, finds its two unstable roots and shows where no drift exists. Section 10.5 describes the measurements and the laws: the grip controller, the driver's nested loops, the scheduled LQR, the friction estimate, the path loop and the reversal. Section 10.6 measures the ten cases, Section 10.7 describes the sensors, actuators and controllers of real drifting cars, Section 10.8 states the limitations and Section 10.9 concludes.

10.2 From the Handling Limit to Automated Drifting

The lateral dynamics of the car were first written as a control problem in the 1950s. Segel's linear model of the automobile's response to the steering wheel, checked against instrumented road tests, made handling a subject of dynamic analysis [9], and its reduction to sideslip and yaw on one track per axle still opens vehicle dynamics texts [10]. Its tires are linear, so such a model stops where the tires begin to slide. The tire models that carry it past that point came from the brush idea, in which the tread elements stick in the front of the contact patch and slide in the rear, so that the lateral force rises linearly with slip, bends over and settles at the friction limit [11].

What happens beyond the limit was studied first as a hazard. A bifurcation analysis of the nonlinear single-track model traced how the stable cornering equilibrium is lost as the tires saturate, and robust front-wheel steering control was designed to keep the car stable [2]. Production cars took the same view. Electronic stability control brakes individual wheels to keep the measured yaw rate and sideslip near a reference computed from the driver's steering, and a widely fitted system had five years of service experience by 2000 [12]. In that design a drift is a fault to be prevented, and the yaw-rate loop of case 2 is its simplest form.

Drifting had long been a driving technique before analysis caught up with it in the late 2000s. The handling characteristics and the stability of the steady powerslide were worked out with the tools of ordinary steady cornering [3]. Equilibrium families were computed against sideslip, phase portraits drawn at fixed speed, and a controller was built that uses the rear drive force as the handle on the saturated rear tire through the friction circle [5]. State feedback designed by LQR on the linearization at a drift equilibrium was shown to stabilize steady-state drifting of a rear-drive car [6]. A controller

Table 10.1 Signals and States of the Drifting Car at the Design Point

Variable	Role	Design	Range
β	CV, sideslip	−35°	SP 0–45°
r	CV, yaw rate	0.927 rad/s	—
ρ	CV of the path loop, distance	8.63 m	SP 7–12 m
V	speed	8 m/s	SP 4–12 m/s
δ_{cmd}	MV, steer command	−25.04°	±40°
T	MV, drive torque	1152 N m (61.9 %)	0–1860 N m
μ_F, μ_R	DV, friction	0.9	0.9 dry, 0.6 wet
kick, gust	DV, events	—	10°, 2 kN for 1 s
τ_T, θ_δ	DV, throttle lag, steer delay	0	0–300 ms, 0–200 ms
U, V_b	states, body velocities	6.55, −4.59 m/s	—
ω, κ	states, driven wheel	38.16 rad/s, 0.770	—
δ	state, road-wheel steer	−25.04°	±40°
x, y, ψ	states, pose on the pad	—	fence
θ	state, rear tire heat	0	0–1

framework for autonomous drifting, with a proof of stability, was validated experimentally on a research test car [7]. Path tracking followed, first with simultaneous stabilization and tracking of basic drifting trajectories [13] and then along a general path, which places drifting inside automated vehicle control beyond the stability limits [8]. Friction had meanwhile become a quantity a vehicle can identify in real time from GPS and motion measurements [14]. The simulator follows this sequence, from equilibrium to linearization to state feedback to path loop, as cases a student can run.

> ⧖ HISTORY **Stability Control Becomes Standard Equipment, 2007 to 2011**
>
> In April 2007 a final rule of the United States highway safety agency established Federal Motor Vehicle Safety Standard No. 126, which requires electronic stability control on light vehicles of 4536 kg gross vehicle weight rating or less. The rule describes the systems as automatic braking of individual wheels that helps the driver when the car begins to lose directional stability at the rear wheels, a spin out, or control at the front wheels, a plow out. Makers had fitted about 29 percent of light vehicles of model year 2006 with it. A phase-in from September 1, 2008 required it on every new light vehicle by September 1, 2011. The system must determine the yaw rate and estimate the sideslip or its rate, and the rule defines sideslip as the arctangent of the lateral over the longitudinal velocity of the center of gravity [4], the β of this chapter.

10.3 The Car and the Pad

The simulated vehicle is a generic rear-drive coupe of 1450 kg on a flat asphalt skidpad, with no manufacturer's data behind it, as shown in Figure 10.1. Its four tires are represented by two equivalent axle contacts, with forces, normal loads and stiffnesses summed over each axle's pair. Its center of mass lies $a = 1.0$ m behind the front axle and $b = 1.5$ m ahead of the rear, and the static loads are $F_{zf} = mgb/L = 8.53$ kN and $F_{zr} = mga/L = 5.69$ kN with $L = a + b$. The pad carries two clusters of nine cones 18 m apart, eight on a 5 m ring and one at the center of each, a painted 9 m circle around each cluster (the two circles touch, which makes the figure-eight), a fence at $-21 \le x \le 39$ m, $|y| \le 19$ m, and a quarter of circle A, from 3 to 16 m out, that can turn wet.

The variables appear in Table 10.1 by their role. The controlled variables are the sideslip and the yaw rate, with the distance from the cluster's center in an outer loop, and the two manipulated variables are the steer command and the drive torque. Ten states are integrated, the position (x, y) and heading ψ on the pad, the body velocities U and V_b, the yaw rate r, the driven wheel's speed ω and transient slip κ, the road-wheel steer δ and a rear tire heat θ. The sideslip is $\beta = \operatorname{atan2}(V_b, U)$ and the speed $V = \sqrt{U^2 + V_b^2}$. A left-hand drift has $r > 0$ and $\beta < 0$, and every target is entered as a positive sideslip.

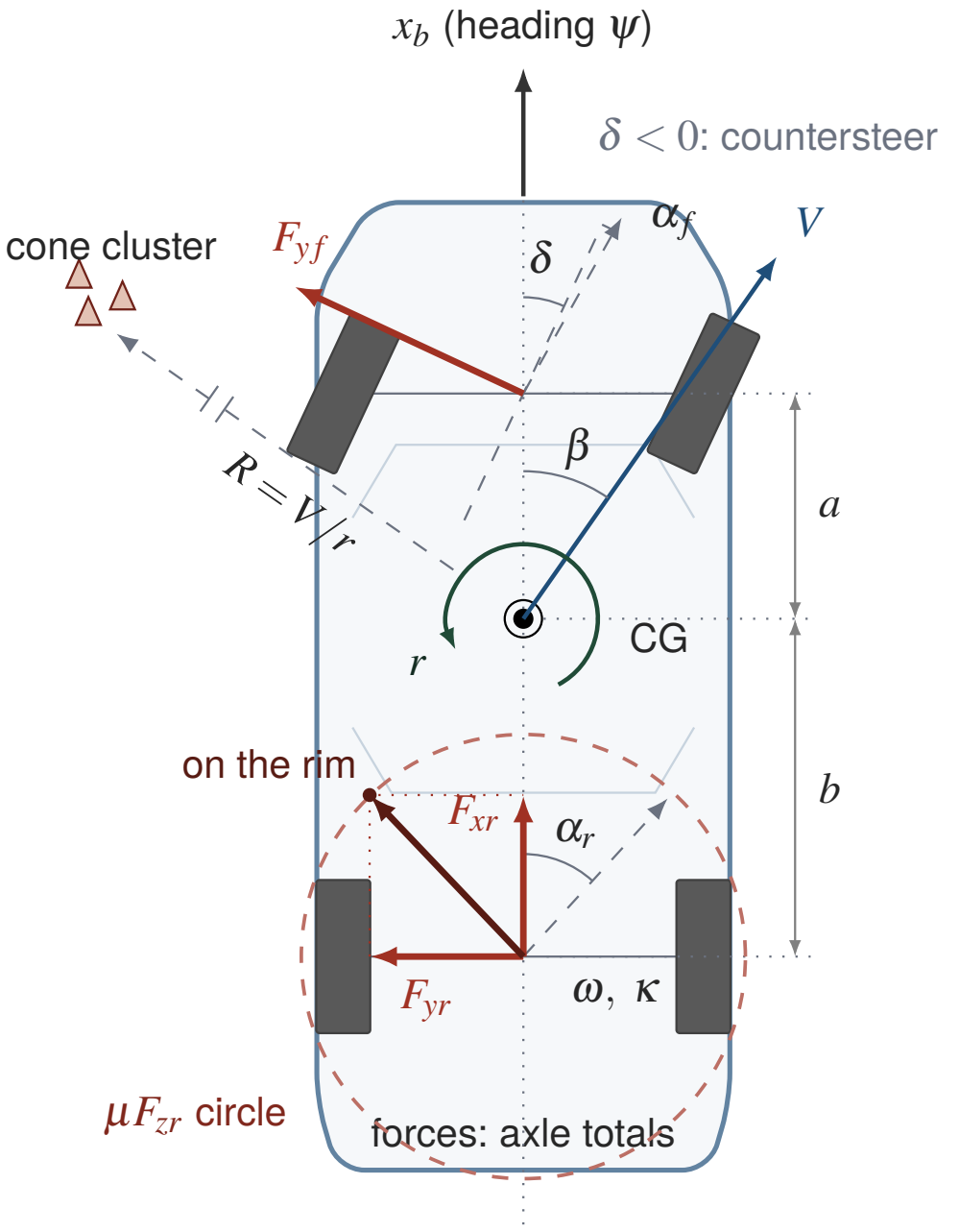


Figure 10.1 The four-tire car with the single-track model's axle-total forces, at the left-hand drift of the design point (35° at 8 m/s). The heading x_b and the velocity V differ by the sideslip β, whose components along and across x_b are U and V_b. Both front tires turn by δ against the turn. Their equivalent axle contact slips at α_f and carries the total lateral force F_{yf} within μF_{zf}. The driven rear pair shares the modeled speed ω and slip ratio κ. Its equivalent contact slips at α_r, and the axle forces F_{xr} and F_{yr} add to a resultant on the circle of radius μF_{zr}. The car yaws at r around a center $R = V/r$ away. Lengths to scale, forces at one scale, the radius broken.

10.3.1 Planar Motion

The body moves in the plane under the tire forces, aerodynamic drag and a possible lateral gust F_g, as shown in Equation (10.1).

$$\begin{aligned} m(\dot{U} - rV_b) &= F_{xr} + F_{xf}\cos\delta - F_{yf}\sin\delta \\ &\quad - \tfrac{1}{2}\rho_a C_d A\, U|U| \\ m(\dot{V}_b + rU) &= F_{yr} + F_{xf}\sin\delta + F_{yf}\cos\delta + F_g \\ I_z\dot{r} &= a(F_{yf}\cos\delta + F_{xf}\sin\delta) - bF_{yr} \\ \dot{x} &= U\cos\psi - V_b\sin\psi, \quad \dot{y} = U\sin\psi + V_b\cos\psi \\ \dot{\psi} &= r \end{aligned} \tag{10.1}$$

where F_{xf} is zero with rear drive and F_{xr} is zero with the front-drive option of case 9. The slip angles of the two contacts are the angles between each wheel's plane and its velocity, as shown in Equation (10.2).

$$\alpha_f = \operatorname{atan2}(V_b + ar,\ \hat{U}) - \delta, \qquad \alpha_r = \operatorname{atan2}(V_b - br,\ \hat{U}) \tag{10.2}$$

where $\hat{U} = \max(|U|, u_{min})$ keeps the angles defined near a standstill. At the design point $\alpha_r = -42.4^\circ$ and $\alpha_f = -4.15^\circ$, so the rear slides while the front grips.

10.3.2 Brush Tires and the Friction Circle

Both tires follow the Fiala brush model [11] from the framework's shared tire module. A tire that carries only lateral force, with cornering stiffness C and limit F, gives Equation (10.3).

$$F_y = \begin{cases} -Ct + \dfrac{C^2}{3F}|t|t - \dfrac{C^3}{27F^2}t^3, & |\alpha| < \operatorname{atan}\dfrac{3F}{C} \\ -F\operatorname{sign}\alpha, & \text{beyond} \end{cases} \tag{10.3}$$

where $t = \tan\alpha$. The undriven axle uses it with $F = \mu F_z$. The force leaves the line $-C\tan\alpha$ at once, reaches the limit with zero slope at the sliding angle, 5.48° for the rear on the dry pad, and stays there. The driven axle carries combined slip from its slip ratio and slip angle, as shown in Equation (10.4).

$$\begin{aligned} s_x &= \frac{C_x\kappa}{1+\kappa}, \quad s_y = \frac{C_\alpha\tan\alpha}{1+\kappa} \\ s &= \sqrt{s_x^2 + s_y^2}, \quad g = \frac{s}{3\mu F_z} \\ F &= \mu F_z\big(1 - (1-g)^3\big)\ \ (g<1), \qquad F = \mu F_z\ \ (g \ge 1) \\ F_x &= F\,s_x/s, \qquad F_y = -F\,s_y/s \end{aligned} \tag{10.4}$$

where C_x is the longitudinal stiffness per unit slip, C_α the cornering stiffness and the denominator $1+\kappa$ is floored at 0.1. The magnitude is the brush curve of the composite slip, so the form reduces to Equation (10.3) at $\kappa = 0$ and to the longitudinal brush curve at $\alpha = 0$, and once $g \ge 1$ the force lies on the circle of radius μF_z. Chapter 4 treats the longitudinal side alone, with a friction curve that peaks and then falls. Here the lateral force on the circle is $F_{yr} = -\sqrt{(\mu F_{zr})^2 - F_{xr}^2}\operatorname{sign}\alpha_r$, which is the friction circle as a control law: every newton of drive the rear carries is paid for in lateral grip. The force is directed along the stiffness-weighted slip (s_x, s_y), a common simplification. The full brush model points a sliding tire's force against its sliding velocity, along $(\kappa, \tan\alpha)$ [11], and at the design point the two directions are 43.48° and 49.85° from the wheel's plane, 6.37° apart. The equilibria below do not depend on the direction, because there the rear force is on its circle and the drive force is solved for directly. The slip ratio that holds it, the linearization and the closed loop do depend on it.

Table 10.2 Model Parameters. H: Hidden, A: Adjustable in the Advanced Panel

Symbol	Meaning	Value	Unit	
m, I_z	mass, yaw inertia	1450, 2300	kg, kg m^2	A
a, b	CG behind front, ahead of rear axle	1.0, 1.5	m	A
h	CG height (load transfer)	0.5	m	A
C_dA, ρ_a	drag area, air density	0.65, 1.2	m^2, kg/m^3	A, H
C_f, C_r	cornering stiffness, front, rear	120, 160	kN/rad	A
C_x	rear longitudinal stiffness	200	kN	A
μ_{dry}, μ_{wet}	friction, dry pad, wet patch	0.9, 0.6	—	A
σ	slip relaxation length	0.3	m	A
v_ε, u_{min}	slip floors	0.3, 0.5	m/s	H
$\Delta\mu_{heat}$	rear friction lost when hot	0.15	—	A
P_{heat}, τ_θ	heating power, time constant	40, 20	kW, s	A
R, J_w	wheel radius, axle inertia	0.31, 2.0	m, kg m^2	A
$F_{x,max}, P_{max}$	drive force, power limit	6, 120	kN, kW	A
$\delta_{max}, \dot{\delta}_{max}$	steer lock, servo rate	40, 400	°, °/s	A
τ_s	steer servo lag	30	ms	A

10.3.3 Driven Wheel, Steering and Drive

The driven wheel has its own speed and a transient slip that relaxes over a length σ, and the steer follows its command through a servo, as shown in Equation (10.5).

$$\begin{aligned} J_w\dot{\omega} &= T_q - RF_{x,drive} \\ T_q &= \min\left(T,\ P_{max}/\max(|\omega|,1)\right) \\ \sigma\dot{\kappa} &= R\omega - u_w - (|u_w| + v_\varepsilon)\kappa \\ \dot{\delta} &= \mathrm{clip}\left((\delta_{cmd} - \delta)/\tau_s,\ \pm\dot{\delta}_{max}\right) \end{aligned} \tag{10.5}$$

where u_w is the driven wheel's own forward speed (U at the rear), so the steady slip is the drive slip ratio $(R\omega - u_w)/(|u_w| + v_\varepsilon)$ and $v_\varepsilon = 0.3\,\mathrm{m/s}$ keeps it finite at rest. The computer clips the torque command to $\min(F_{x,max}R,\ P_{max}/\max(|\omega|,1))$, 1860 N m or the 120 kW power limit, and the plant applies the power limit again continuously, so the delivered power never passes 120.0 kW. The electric drive and its 2.0 kg m^2 axle make the throttle a fast torque source.

Two options are off in every case. Longitudinal load transfer moves hF_x/L onto the rear under drive, with the center of mass $h = 0.5\,\mathrm{m}$ high. Tire heat follows $\tau_\theta\dot{\theta} = P_{slide}/P_{heat} - \theta$, with P_{slide} the rear's force times its slip speed while it slides, and lowers the rear friction by $\Delta\mu_{heat}\min(1, \max(0, \theta))$.

10.3.4 Parameters, Numerics and Events

Table 10.2 gives every parameter. All are teaching values in the range the drifting literature reports [5–7], not any paper's vehicle. Physics runs on the framework's integrator (Section 1.6.1) with a step of 1 ms. The stiffest pair is the driven wheel against its slip, and halving the step leaves every case's fate unchanged. The growth rate of case 3 is 1.982/s at both steps, the recovery of case 5 0.914 s at both, and the challenge's scores 0 and 58 at both. Each case starts either on the drift equilibrium of its targets, placed on its circle, or on a grip turn with the kinematic steer, and a drift start is an equilibrium of the integrated model to a residual below 10^{-9}. Hard events live outside the derivatives. SPUN latches when $|\beta|$ passes 70°, with the crossing located inside the step, GRIPPED OUT when the rear has not slid for 2 s under a drift command of 10° or more outside a reversal, OFF PAD at the fence, and a cone falls when the body rectangle, grown by the cone's 0.15 m radius, touches it. These thresholds are teaching values. Real stability control works from yaw-rate and sideslip references set by the vehicle's own calibration [12], and friction values on real pavement come from tire testing [11].

10.3.5 Assumptions

The single-track model lumps each axle into one tire, so there is no lateral load transfer, no roll and no differential. A real drift car loses most of the inside rear's load and depends on a locked differential to keep both rear wheels sliding, and the model holds best where the two rear tires behave alike. The tires have constant stiffness and friction, no camber and no lateral relaxation, and the brush curve keeps its peak force while sliding, whereas a real tire's sliding friction usually falls below its peak. The combined-slip direction is the stiffness-weighted simplification described above. Axle loads are static unless load transfer is switched on. The drive is an electric torque source with a lumped axle. The pad is flat and its friction is uniform except for the wet quarter, whose edge is a step. The model is least reliable at very low speed, where the slip floors act, and at the moment a tire changes between sliding and gripping, where the brush curve's corner at the sliding angle meets a real tire's rounded peak.

10.4 The Drift as an Equilibrium

10.4.1 Solving for the Steady Drift

For a speed V and a sideslip β the car drifts in steady state when $\dot{U}$, $\dot{V}_b$ and $\dot{r}$ vanish together. With $U = V\cos\beta$ and $V_b = V\sin\beta$ fixed, Equation (10.1) gives three residuals in the three unknowns δ^*, r^* and F_{xr}^*, as shown in Equation (10.6).

$$\begin{aligned} 0 &= F_{xr} - F_{yf}\sin\delta - \tfrac{1}{2}\rho_a C_d A\, U^2 + m r V_b \\ 0 &= F_{yf}\cos\delta + F_{yr} - m r U \\ 0 &= a F_{yf}\cos\delta - b F_{yr} \end{aligned} \tag{10.6}$$

with F_{yf} from Equation (10.3) at μF_{zf} and the rear on its circle, $F_{yr} = \mathrm{Fiala}\big(\alpha_r, C_r, \sqrt{(\mu F_{zr})^2 - F_{xr}^2}\big)$. A damped Newton iteration with a numerical Jacobian solves it from three starting guesses, with the drive force written as $\mu F_{zr}\tanh(z/\mu F_{zr})$ so that it can only be a force the tire can give. A second Newton pass on the integrated model's own rates then finds the slip ratio κ^*, wheel speed ω^* and torque T^* that hold F_{xr}^* at α_r, so the state returned is an equilibrium of Equations (10.1) to (10.5) to round-off. At the design point, 35° and 8 m/s on the dry pad, the solution is $\delta^* = -25.04°$, $r^* = 0.927\,\mathrm{rad/s}$ (a radius of 8.63 m), $F_{xr}^* = 3716\,\mathrm{N}$, $\kappa^* = 0.770$ and $T^* = 1152\,\mathrm{N\,m}$, 44.0 kW of drive.

The family of these equilibria at 8 m/s is shown in Figure 10.2a. Between 5° and 45° the drive force rises from 2.10 to 4.17 kN and the radius from 7.92 to 8.84 m, while the steer passes from 8.5° into the turn to 36.2° against it, crossing zero between 12° and 13°. Countersteer is therefore a consequence of large sideslip rather than its definition. A deeper drift needs more drive, because the rear's lateral share must fall as the front's steer angle rises. On the wet pad the design target moves to $\delta^* = -27.88°$, $r^* = 0.647\,\mathrm{rad/s}$, a radius of 12.37 m and 2369 N, which is why a design fixed on the dry pad's numbers fails on the wet patch of case 6.

10.4.2 Two Unstable Roots

The model is linearized about the equilibrium by central differences of its rate function, as in Section 1.4.4, in the six states $\tilde{x} = [U, V_b, r, \omega, \kappa, \delta] - x^*$ and the two inputs $\tilde{u} = [\delta_{cmd}, T] - u^*$, giving $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$. At the design point A has the eigenvalues $+1.955$, $+0.206$, -8.55, $-11.60 \pm 15.85i$ and -33.33 1/s, the last being the steer servo's $1/\tau_s$. Two roots are unstable, at every sideslip from 5° to 45°, as shown in Figure 10.2b. The faster, the slide, doubles an error every 0.355 s and peaks at 1.956/s near 37°. The slower is the speed drifting under a held torque. A three-state car written from the same tire functions, with (U, V_b, r) as states and the steer and drive force held, has both, 1.966 and 0.215/s at the design point, while its lateral pair (V_b, r) at a fixed speed has only one, 1.889/s. That single root is the one an analysis at fixed speed reports [5], and the second appears only when

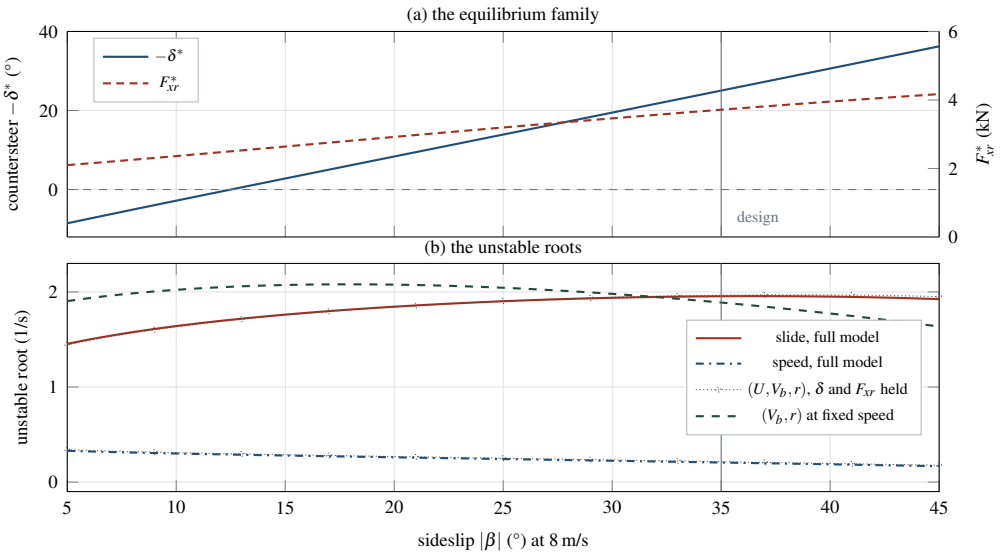


Figure 10.2 The drift equilibrium family at 8 m/s on the dry pad, from the simulator's own solver and linearization. (a) Countersteer (solid) and rear drive force (densely dashed) against sideslip. The steer points into the turn below 12.5°. (b) The unstable roots. The full model has two at every sideslip, the slide (solid) and a slow speed mode (dash-dotted), and the three-state car with steer and drive force held reproduces both (dotted with circles). Holding the speed fixed leaves one (dashed). The vertical line marks the design point.

the speed is free. At 10 m/s the pair moves to 2.284 and 0.100/s, and on the wet pad at 8 m/s to 1.991 and 0.090/s.

Because the rear is saturated, its lateral force cannot respond to its slip angle, and the yaw moment that would restore a gripping car is missing. The linearization says how the two inputs act instead. Steering left raises the yaw acceleration through the gripping front, and more torque spins the wheel up, which moves the rear force along its circle toward drive and away from lateral grip, so the rear swings out further. Both signs are checked by the app's own tests.

🕮 **Pause.** Try Exercise 10.1 before reading on.

10.4.3 Where No Drift Exists

Case 9 drives the front axle and lets the rear roll. With the rear saturated and undriven, $|F_{yr}| = \mu_R F_{zr}$, and the yaw balance of Equation (10.6) asks the front for a body-lateral force $(b/a)\mu_R F_{zr}$. With static loads this is $\mu_R mgab/(aL) = \mu_R F_{zf}$. When both axles see the same friction, that is the front's whole friction circle, so the front can carry no body-longitudinal force. The longitudinal balance then needs $mr|V_b| + \frac{1}{2}\rho_a C_d A\, U^2 = 0$, which no drift can satisfy, because both terms are positive. The result holds for any weight distribution, and load transfer under forward drive only lowers F_{zf} against the same demand. When the front has more grip than the rear, the lateral balance gives $rU = \mu_R g$ (drag neglected), the front needs a longitudinal force $m\mu_R g \tan|\beta|$ beside its lateral $\mu_R F_{zf}$, and its circle allows a saturated-rear drift only up to

$$\tan|\beta| \le \frac{b}{L}\,\frac{\sqrt{\mu_F^2 - \mu_R^2}}{\mu_R}. \tag{10.7}$$

For $\mu_F = 0.9$ on $\mu_R = 0.6$ the bound is 33.9°, and the app's solver finds such drifts from 2° to 33° at 8 m/s. With equal friction the bound is zero. Over 1230 targets, every 0.5 m/s from 5 to 12 m/s and

every degree from 5° to 45° on both surfaces, the front-drive solver never converges, and its smallest residual is 0.0343 m/s^2. The rear-drive solver converges at 1228 of the same targets with residuals below 2×10^{-11}. The two exceptions, cold starts at low speed and small sideslip, are discussed in Section 10.8.

Pause. Try Exercise 10.3 before reading on.

10.5 Implemented Control System

Every law runs every 10 ms and reads only a measurement packet. A dual-antenna navigation unit gives the speed (± 0.03 m/s), the sideslip ($\pm 0.15°$), the position (± 2 cm) and the heading ($\pm 0.1°$), a gyro the yaw rate ($\pm 0.2°/$s), an accelerometer the lateral acceleration (± 0.05 m/s^2), and encoders the wheel speed (± 0.05 rad/s) and the steer ($\pm 0.05°$). Each noise is colored with a 20 ms correlation time and seeded, so a run replays bit for bit (Section 1.6.1). The body velocities follow as $U = V\cos\beta$ and $V_b = V\sin\beta$, and the transient slip, which no instrument measures, is replaced by its quasi-steady value $\hat{\kappa} = (R\omega - U)/(|U| + v_\varepsilon)$. The surface, the tire forces, the true slip, the heat and any gust never reach a law. Each law is written for a left-hand drift in direction-normalized signals (the sideslip $b = -s\beta$, the yaw rate sr, the steer $s\delta$, with $s = \pm 1$), so one law serves both directions.

10.5.1 The Grip Controller

The grip controller is the stability-control pattern of the lane-keeping simulator of Volume 20, Chapter 2. A proportional-integral (PI) loop steers toward a yaw-rate target, and a second PI loop sets the torque from the speed error, as shown in Equation (10.8).

$$r_{ref} = \frac{V_{SP}}{R_{path}} + K_{\beta r}\max\big(0,\ \beta_{SP} - \max(b,0)\big) \tag{10.8}$$

where the second term asks for extra yaw rate while sideslip is missing. The request is one-sided because a two-sided one added yaw rate on grip, where b is negative, and drove the car into the cones. The defaults are $K_g = 0.5$ and $\tau_{I,g} = 0.5$ s on the steer, $K_v = 600$ N m per m/s and $\tau_{I,v} = 2$ s on the torque, and $K_{\beta r} = 1.0/$s.

10.5.2 The Driver's Nested Loops

Two PI loops reproduce the way a driver holds a slide, each on top of the dry pad's equilibrium as a feedforward, as shown in Equation (10.9).

$$T = T^* + \mathrm{PI}_\beta(\beta_{SP} - b), \qquad \delta_{cmd} = \delta^* + \mathrm{PI}_r(r^* - sr) \tag{10.9}$$

The throttle holds the sideslip because more drive takes lateral force from the rear, which slides out further, and the steer holds the yaw rate. Each loop moves the other's variable. The reference tuning is $K_\beta = 104.7$ N m/° (6000 N m/rad), $\tau_{I,\beta} = 0.5$ s, $K_r = 1.0$ and $\tau_{I,r} = 0.5$ s. Case 4 and the challenge ship a loose tuning with half the gains and twice the integral times.

Every loop is the series' velocity-form PID of Section 1.6.2, with back-calculation on the integral alone and a tracking time equal to the integral time τ_I. Its output is clamped to 0 to the torque limit and to $\pm 40°$ of steer. Every loop not in charge tracks the applied commands, so Manual to Auto and any change of law are bumpless.

10.5.3 LQR About the Equilibrium

The LQR acts on the whole measured state, as shown in Equation (10.10).

$$\begin{aligned} u &= u^* - K(\hat{x} - x^*), \qquad J = \sum_k \tilde{x}_k^{\mathsf{T}} Q \tilde{x}_k + \tilde{u}_k^{\mathsf{T}} R_u \tilde{u}_k \\ K &= (R_u + \Gamma^{\mathsf{T}} P \Gamma)^{-1} \Gamma^{\mathsf{T}} P \Phi \end{aligned} \tag{10.10}$$

where Φ and Γ are the zero-order-hold discretization of (A,B) at 10 ms and P solves the discrete algebraic Riccati equation (Section 1.4.5). The series' Riccati module solves the single-input regulator, so the two-input equation is solved by its Kalman routine on the dual system $(\Phi^{\mathsf{T}},\Gamma^{\mathsf{T}})$, whose a-priori covariance satisfies the same equation [15]. The weights are diagonal by Bryson's rule, $Q=\mathrm{diag}(1/s_i^2)$ with scales of 1 m/s on U, $V\beta_s$ on V_b ($\beta_s=2^\circ$), 0.1 rad/s on r, 20 rad/s on ω, 1 on κ and 0.5 rad on δ, and $R_u=\mathrm{diag}(1/\delta_s^2,1/T_s^2)$ with $\delta_s=11.5^\circ$ and $T_s=800\,\mathrm{N\,m}$. At the design point the gain's steer row is $(-0.187,\ 0.008,\ 1.589,\ 0.002,\ 0.054,\ 0.724)$ and its torque row $(29.6,\ -2381,\ 4647,\ 45.0,\ 153.6,\ 349.9)$ in SI units, and the closed loop's roots are -0.81, -7.54, $-19.84\pm18.28i$ and $-33.06\pm20.64i$ 1/s. An independent Riccati recursion reproduces the gain to 2.6×10^{-8}.

Rather than solve the Riccati equation at every control period, the gain is scheduled, here on speed and friction where Chapter 9 schedules its LQR on speed alone. Nodes sit every 1 m/s from 4 to 14 m/s and every 0.1 of friction from 0.3 to 1.3, each designed at the exact equilibrium of its node, and the gain is interpolated bilinearly between the four nodes around the target. The equilibrium itself is solved exactly at the target speed (on a 0.05 m/s grid) and at the friction the schedule names, either the dry pad's 0.9 or the estimate $\hat{\mu}$ rounded to 0.02. When the LQR takes over from another law it adds the last command's difference as a bias that decays with a 0.5 s time constant. Where the solver finds no equilibrium, as in case 9, the LQR hands the car to the nested loops without feedforward.

10.5.4 Friction Estimate, Path Loop and Reversal

While the rear slides its force is on its circle, so its magnitude over its load is the friction. The lateral and yaw balances give the rear lateral force from the inertial measurements, and the drive force follows from the torque command less the wheel's spin-up, as shown in Equation (10.11).

$$\begin{aligned}\hat{F}_{yr}&=\frac{a\,m\,\bar{a}_y-I_z\bar{\dot{r}}}{L},\qquad \hat{F}_{xr}=\frac{T-J_w\bar{\dot{\omega}}}{R}\\ \hat{\mu}_{inst}&=\sqrt{\hat{F}_{xr}^2+\hat{F}_{yr}^2}\,\big/\,F_{zr}\end{aligned}\tag{10.11}$$

where the bars are 50 ms filters (the acceleration's started from its first reading). The estimate $\hat{\mu}$ follows $\hat{\mu}_{inst}$, clipped to 0.2–1.3, through a 150 ms filter only while the rear's slip speed exceeds 1 m/s, and holds otherwise, starting from 0.9. With front drive $\hat{F}_{xr}=0$. The idea follows the friction identification of motion-based estimators [14], simplified to a saturated tire.

The path loop trims the target speed on the distance ρ from the cluster's center and its rate, $V_T=V_{SP}-K_{p,\rho}(\rho-R_{path})-K_{d,\rho}\dot{\rho}$, clipped to $V_{SP}\pm2.5\,\mathrm{m/s}$, with $K_{p,\rho}=0.5\,(\mathrm{m/s})/\mathrm{m}$, $K_{d,\rho}=1.0\,\mathrm{s}$ and $R_{path}=9\,\mathrm{m}$. Because the equilibrium's radius grows with its speed (8.63 m at 8 m/s and 13.12 m at 10 m/s), choosing the speed chooses the circle, which is the idea of tracking a path by choosing among drift equilibria [13] in its simplest form. The figure-eight reversal is open loop. 10° of arc before the circles' common point the throttle is lifted and the steer goes to full opposite lock until the yaw rate has reversed past 1.5 rad/s, then 1200 N m is applied with the lock held until the new side's sideslip passes 5°, and the new side's law takes over. A transition counts once the sideslip has stayed 0.5 s in the new band within 5 s of the flick.

The application keeps three models apart. The plant is everything in Section 10.3. The identified model is the growth rate a learner reads from the trend of case 3. The controller's model is the solver's and the linearization's copy of the declared parameters, which knows neither the wet patch nor a gust, and which meets the true friction only through $\hat{\mu}$.

10.6 Ten Cases, Measured

The cases appear in Table 10.3. All share the reference settings (the LQR scheduled on $\hat{\mu}$, the path loop on, 35° at 8 m/s) and depart from them as their rows say. Each number below is taken from the simulator's engine run without a display at the seed it ships with, advanced in the same 1 ms steps the

Table 10.3 The Ten Cases

#	Case and settings	Lesson
1	Manual on the drift, stations at its inputs	the drift is unstable
2	Grip controller asked for 30°, started on grip	grip control at the limit
3	No controller, started on the equilibrium	the unstable root
4	Loose nested loops, speed target 8 to 10 m/s at 10 s, 30 s	two loops that fight
5	LQR on the dry schedule, 10° kick at 8 s	both actuators together
6	LQR on the dry schedule, path loop, wet quarter from 0, 30 s	the equilibrium moves
7	Figure-eight from 5 s	the reversal
8	LQR, path loop off, circle 2.5 m off center	the path loop
9	Front drive	no equilibrium exists
10	Challenge: loose nested loops, figure-eight from 30 s, wet from 25 s, 60 s	the whole problem

browser uses, with each named setting applied before the first step and a run length of 20 s unless a row names another. "In band" counts the seconds with the true sideslip within 5° of its target outside a reversal. Every case loads identically when chosen first, from the list after another case, or after Reset (Section 1.6.4). Figure 10.3 draws the time in band of every run that reports one.

10.6.1 By Hand and With No Controller

Case 1 starts in the drift with the steer and throttle stations at the equilibrium's inputs rounded to their resolution, −25.0° and 61.9 %, which leaves them 0.041° and 0.029 % away. Untouched, the car is 5° off at 2.98 s and spins at 4.29 s. Flooring the throttle spins it at 0.97 s, and closing it lets the rear grip again, GRIPPED OUT at 2.86 s, after which the car turns the other way with 13° to 14.5° of sideslip. The nested loops in Auto hold the band for 19.99 of the 20 seconds. Case 3 is the same start with no law. The sideslip error passes 0.5° at 1.82 s and 5° at 2.98 s, a factor of ten in 1.16 s and a growth of 1.98/s against the root's 1.95/s. With the inputs set exactly before Run the car stays on the equilibrium for the whole run, because the simulated state is then a fixed point to the last bit, and the Nudge event (1° of velocity) at 2 s takes it 5° off at 3.22 s. A real car always carries that nudge. The other targets of the family leave as fast, at 3.16 s from 25°, 3.13 s from 45° and 3.54 s from 35° at 10 m/s.

> **TRY IT IN THE APP** **3 · The Equilibrium, With No Controller**
>
> Load the scenario, press Run without touching anything and watch the sideslip badge. The car starts on its drift equilibrium, yet the sideslip error passes 0.5° at 1.82 s and 5° at 2.98 s, a factor of ten in 1.16 s. That growth is the slide's unstable root, read off a trend.

10.6.2 Grip Control and the Two Loops

The grip controller of case 2 steers into the turn for more yaw rate, the rear breaks away and the car spins at 3.61 s. Every request of 10° or more spins it, at 9.91 s for 10°, 5.62 s for 15° and 3.53 s for 35°, and a larger request gain spins it sooner (3.41 s at $K_{\beta r} = 2$). With the request at zero it holds the 9 m circle on grip with 7.23° to 7.30° of sideslip, the nose outside the velocity and no cone touched. Its loops assume a tire whose force rises with slip, which the drift does not have.

Pause. Try Exercise 10.2 before reading on.

The loose nested loops of case 4 hold 34.7° to 35.4° until the speed target steps to 10 m/s at 10 s. After it they pass the disturbance back and forth in a swing with a 3.26 s period that grows, leaving the band at 17.23 s and spanning 25.4° to 44.6° by 30 s, as listed in Table 10.4. Doubling either gain alone stops the swing. With the reference tuning the sideslip stays within 32.2° to 36.9° and is back within 2° for good at 10.76 s, and the LQR on the same step keeps 34.9° to 38.9°.

Table 10.4 Measured Results at the Default Seed

Case	Law	In band (s)	Cones	Outcome
1	by hand, untouched	—	1	SPUN 4.29 s
	nested, Auto	19.99	0	held
2	grip, 30°	—	2	SPUN 3.61 s
4, 30 s	nested, loose	24.0	0	swing 25.4–44.6°
	nested, reference	—	0	32.2–36.9°
5	LQR, dry	19.7	0	±2° in 0.92 s
	nested, reference	18.5	4	peak 42.1°, 3.74 s
	nested, loose	11.0	4	peak 47.5°
6, 30 s	LQR, dry schedule	3.9	3	SPUN 10.06 s
	LQR, $\hat{\mu}$ schedule	30.0	0	32.8–39.9°
	nested, reference	12.0	—	on the pad
7	LQR, $\hat{\mu}$, path	16.6	0	2 transitions
	nested, reference	10.1	5	1 transition
	LQR, path off	17.1	3	2 transitions
8	LQR, path off	—	4	ρ 6.08–11.19 m
	LQR, path on	—	0	ρ 8.81–8.82 m
9	LQR (front drive)	—	—	OFF PAD 3.18 s
	grip (front drive)	—	—	GRIPPED OUT 2.30 s
	LQR, rear drive	20.0	0	held

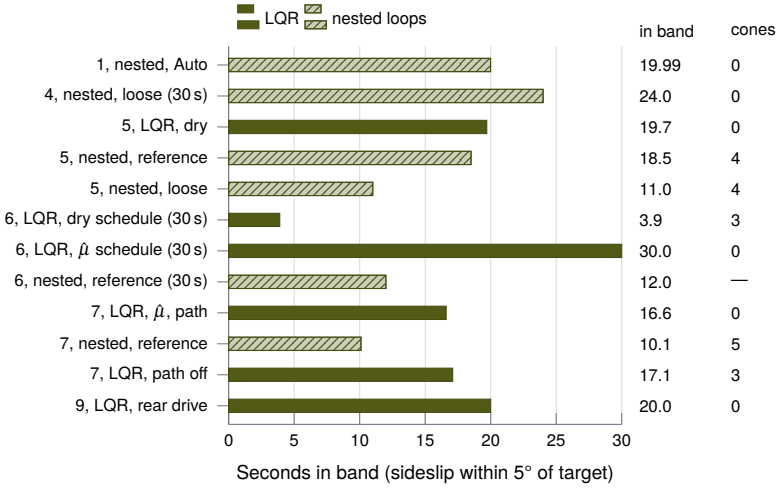


Figure 10.3 The rows of Table 10.4 that report time in band, with their cones. Solid bars are LQR runs, hatched bars nested-loop runs. Runs last 20 s except those marked 30 s. Notice that on the wet patch of case 6 the same LQR holds the band for all 30 s when its schedule follows $\hat{\mu}$ and for only 3.9 s on the dry schedule.

10.6.3 LQR, a Kick and the Wet Patch

The LQR of case 5 brings the sideslip back within 2° of 35° 0.92 s after the 10° kick and never overshoots, reaching 35.04° at most. The reference nested loops overshoot to 42.1°, take 3.74 s and knock four cones. The LQR moves steer and throttle together the moment the kick arrives, because its gain couples every state to both inputs, while each nested loop waits for its own variable to move. In case 6 the wet quarter is on from the start. The LQR on the dry pad's schedule spins the car at 10.06 s. Scheduled on $\hat{\mu}$, the equilibrium and the gain follow the friction, and the car holds 32.8° to 39.9° and 8.62 to 10.10 m from the cluster for all 30 s. The estimate falls to 0.59 on the patch, stays within 0.016 of the true friction on average while the rear slides, and is up to 0.30 off for a moment at the patch's edges, where the friction steps. Without the path loop the fixed design survives but

Table 10.5 The Challenge Over 60 s, Default Seed

Configuration	End	Score
nested, loose, no path loop (shipped)	SPUN 32.54 s	0
LQR, $\hat{\mu}$ schedule, path loop (suggested)	60 s, 2 transitions	58
LQR, dry schedule, path loop	SPUN 37.41 s	0
LQR, $\hat{\mu}$ schedule, no path loop	OFF PAD 46.98 s	0
nested, reference, path loop	60 s, 11 cones	0
nested, reference, no path loop	60 s, 7 cones	0

wanders and knocks eight cones.

> **TRY IT IN THE APP** **6 · The Wet Patch**
>
> Press Run and watch the car each time it crosses the wet quarter. On the dry pad's schedule it spins at 10.06 s. Press Reset, set Friction in the schedule to Estimated $\hat{\mu}$ and run again. The sideslip now stays between 32.8 and 39.9° for all 30 s, while $\hat{\mu}$ falls to 0.59 on the patch.

10.6.4 The Figure-Eight and the Path Loop

Case 7 flicks at 8.37 s and 15.81 s, and each reversal takes 1.13 s and 1.14 s from the flick to the handover. The run completes both transitions, keeps the band for 16.6 of the 20 seconds and touches no cone. The rear tire leaves its friction circle in each reversal, its force falling to 78 % of the limit in the first and 84 % in the second, as shown in Figure 10.4. While the throttle is lifted and the lock is opposite, the rear slip angle swings through zero, from −23.3° at 8.80 s to 2.0° at 9.00 s, and for an instant the rear grips. That instant is what lets the nose cross to the other side. The reference nested loops complete one of the two reversals and knock five cones. In case 8 the car starts on a circle 2.5 m off center. Without the path loop it passes between 6.08 and 11.19 m from the cluster and knocks four cones. With it the distance is within 0.5 m of 9 m from 2.08 s and between 8.81 and 8.82 m from 15 s.

10.6.5 Front Drive and the Challenge

With front drive (case 9) the solver finds no equilibrium, and the LQR falls back to the nested loops. They floor the throttle, the front plows, the rear stops sliding (GRIPPED OUT at 2.29 s) and the car slides wide off the pad at 3.18 s. The grip controller grips out at 2.30 s. No held pair of steer and throttle keeps the rear sliding near 35° either. Of 187 pairs, 147 leave the pad, 39 grip out and one runs on with no time in the band. The same car with rear drive, from the same Reset, holds the band for all 20 s.

> **DID YOU KNOW?** **Front Drive Cannot Hold This Drift**
>
> From the simulator: with front drive (case 9) the solver finds no drift, and the LQR's fallback leaves the pad at 3.18 s. The same car with rear drive holds the band for all 20.0 s (Table 10.4). Which axle is driven decides whether the drift exists at all.

The challenge asks for 60 s of donut and figure-eight with the wet quarter from 25 s, scoring the seconds in band plus the transitions, and zero on a spin or a cone. As shipped, the loose nested loops hold the band until the wet quarter reaches the car. They leave it at 25.28 s, the sideslip peaks at 50.9°, the first cone falls at 27.15 s, and the car spins at 32.54 s, before the figure-eight begins. The suggested settings, the LQR scheduled on $\hat{\mu}$ with the path loop, hold the band for 56.2 s, complete both transitions and touch no cone, a score of 58. Table 10.5 shows that each part is needed. Over seeds 1 to 20 the shipped settings spin between 32.37 s and 32.58 s every time, and the suggested ones pass every time with 56.09 to 56.42 s in the band. At five times real speed the suggested 60 s run plays in 12.1 s of wall time in a desktop browser.

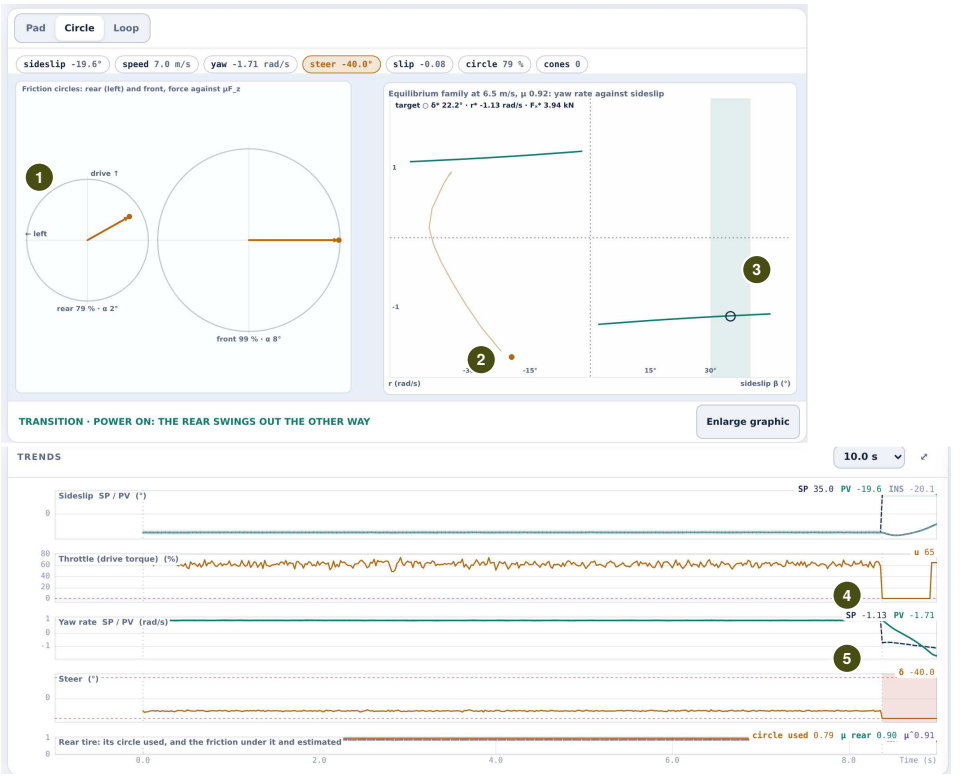


Figure 10.4 The first reversal of the figure-eight (case 7) at $t = 9.00\,\mathrm{s}$ on the Circle view, under the LQR scheduled on $\hat{\mu}$ with the path loop. The picture joins two parts of the app's screen, the plant view above the trends. (1) The rear tire's force has fallen inside its friction circle, to 79 % at a slip angle of 2°, while the front works at 99 %. (2) The state (dot, with its trail) crosses from the left-hand branch of the equilibrium family toward (3) the right-hand target, computed at the path loop's speed target of 6.46 m/s (the plot's title rounds it) and the schedule's friction 0.92 ($\delta^* = 22.2°$, $r^* = -1.13\,\mathrm{rad/s}$, $F_{xr}^* = 3.94\,\mathrm{kN}$). The badges read −19.6° of sideslip, 7.0 m/s, −1.71 rad/s and full opposite lock, −40°, with (4) the throttle back on at 65 % in the power phase. The 10 s trend window shows 35° held from the start, the lift at the flick (8.37 s) and (5) the yaw rate reversing.

10.7 Technology in Practice

What Measures It

A drift controller needs the sideslip and the yaw rate. The yaw rate comes from a gyro, and in common cars the gyro and the lateral accelerometer are already on board for the stability system [16]. The sideslip is harder. Production systems estimate it with extended or unscented Kalman filters, the industrial state of the art [17], while drifting test cars measure it with a satellite-aided inertial unit, as the chapter's packet does with its navigation unit, gyro, accelerometer and encoders.

> **TECHNOLOGY Sensor: Satellite-Aided Inertial Navigation for Sideslip**
>
> Sideslip has no inexpensive direct measurement. An inertial navigation system aided by satellite positioning gives position to the centimeter and measures the slip angle directly [16], and one drifting test car read its state from such a unit at 100 Hz [18]. Trees, tall buildings, mountains and tunnels interrupt the satellite signal, and the sideslip then turns unstable and unreliable [16]. An optical speed sensor gave one test car its reference sideslip, to within $\pm 0.2^\circ$ [17]. The chapter's navigation unit reads it to $\pm 0.15^\circ$.

What Moves It

The two handles are the steer and the drive torque. Experimental controllers that drift along a path ran on a research car whose independent electric motors acted like a fully locked differential, a layout uncommon in standard cars [19]. A production car with a combustion engine couples its rear wheels through a differential and delivers engine torque with a delay, so it cannot set each wheel's speed [18]. The chapter's drive is the research car's kind: an electric torque source on one lumped axle, limited to 1860 N m and 120 kW. Its steer servo follows its command with a 30 ms lag, limited by its rate and its lock.

> **TECHNOLOGY Final Element: A Steering Actuator at Its Limit**
>
> On one production car the electric power steering motor, built for low-torque assistance such as automatic lane changes, gives at most about 30 N m. In an automated drift the steer stayed on its limit and the car spun, most likely because the motor could not overcome the front tires' self-aligning moment, so a driver steered while the controller set the drive torque [19]. On another production car the steering lagged its command by a mean phase delay of 140 ms, and sideslip targets above 0.65 rad were unreliable because the steer command regularly passed the 34° lock, leading to spin-outs [18]. The chapter's servo has a 30 ms lag, a 400 °/s rate and a 40° lock.

Where the Controller Runs

On one test car the controller's generated code ran on a rapid-prototyping computer that also bridged the vehicle's data buses, where measured and estimated states arrived every 10 ms. A nonlinear predictive controller solved its problem at 50 Hz over 25 steps and sent its inputs through the car's driving-assistance interface, which safety features limited [19]. On another, an onboard computer ran the drift controller at a fixed 100 Hz and commanded the production actuators over the vehicle bus [18]. The chapter's laws run every 10 ms, the same 100 Hz, on integration steps of 1 ms. The block that holds its structure is the schedule: a table of LQR gains, each designed at the exact equilibrium of its node and interpolated at the target. Figure 10.5 draws the loop on test-car hardware.

What Is Modernizing It

Three developments carry the chapter's ideas toward production cars. The first is predictive control on a standard passenger car. A nonlinear predictive controller tracking equilibria computed offline brought such a car to 30° of sideslip and held it for several rounds on a dry surface and on a watered skidpad, with a driver steering and the controller setting the drive torque [19]. The second is latency compensation: a state predictor that phase-aligns steer and torque let a combustion-engined sports car hold circular and figure-eight drifts [18]. The third is learning. Agents trained in simulation discover drifting strategies without detailed tire models, but most results remain in simulation because transfer to the real car, robustness to model mismatch and training data are still hard, and a learned diffusion model that needs little data runs on a high-performance processor [18]. The chapter's friction schedule is the simplest form of the adaptation all three point to.

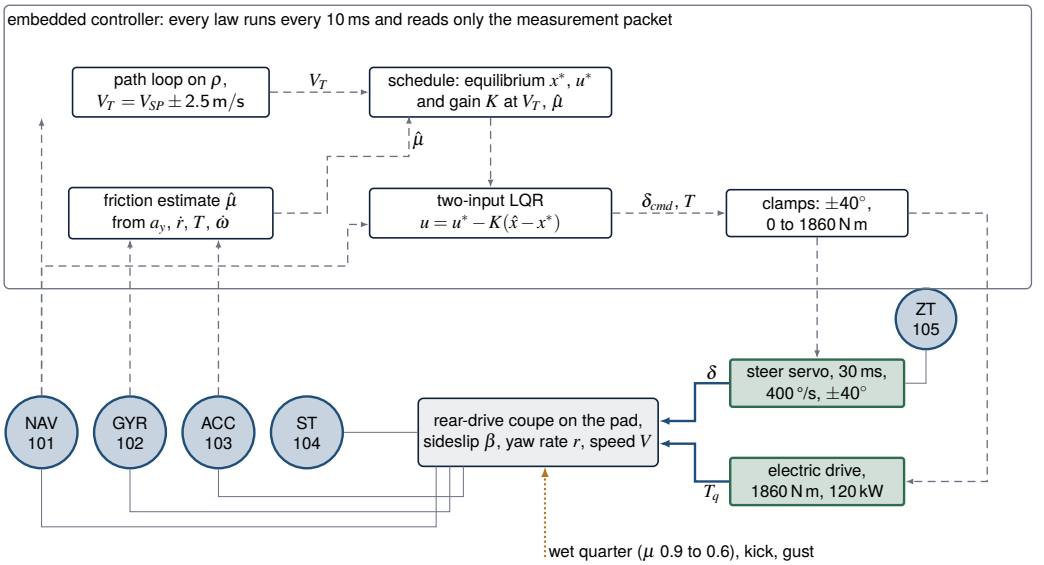


Figure 10.5 The drift loop on test-car hardware. The navigation unit NAV-101 gives the speed, sideslip and pose, the gyro GYR-102 and accelerometer ACC-103 the yaw rate and lateral acceleration, and the encoders ST-104 and ZT-105 the wheel speed and steer. The friction estimate and the path loop's speed target pick the equilibrium and the gain from the schedule, and the LQR sends both inputs, clamped, to the steer servo and the rear electric drive. Notice that one law moves both actuators at once, and that the friction, which no instrument reads, reaches it only through $\hat{\mu}$.

> **IN PRACTICE** **How a Stability System Proves It Stops a Spin**
>
> The 2007 rule adds a performance test. In the sine with dwell maneuver the car coasts at an initial 50 miles per hour while a steering machine turns the wheel through a fixed pattern, repeated at rising steering amplitudes, a maneuver severe enough to spin most cars without the system. The car passes if its yaw rate falls to no more than 35 percent of its peak one second after the wheel returns to straight ahead, and to 20 percent after 1.75 seconds. A responsiveness criterion keeps a car that barely turns from passing [4]. The test checks that a car stops rotating, the opposite of the slide this chapter holds on purpose.

10.8 Limitations

The car is a single track with brush tires at constant friction, so it cannot represent the inside rear wheel unloading, a differential, a tire's fall from peak to sliding friction, or camber. The combined-slip force points along the stiffness-weighted slip rather than the sliding velocity, 6.37° apart at the design point. At the same slip the drive force would be 3302 N instead of 3716 N, so the slip ratio, the linearization and the closed loop would move while the equilibria would not. The wet patch is a step in friction, which the estimate follows with errors up to 0.30 at its edges. The drive is an ideal electric torque source. The reversal is a fixed open-loop sequence tuned for the declared car at 35°, not a planned trajectory, and the challenge's thresholds and the loss rules are teaching values.

The controller's equilibrium is started cold at each target, and at low speed and small sideslip the equations also admit a mirror-image root that turns the other way. The solver accepts only a root whose yaw rate turns the commanded way, and the controller treats a solution past the 40° lock as no equilibrium. Over learner-settable targets (4 m/s to 12 m/s, 1° to 45°) it finds the commanded drift at all 1485 on the dry pad, and on the wet pad at all but five at 45° and 11 m/s or above, whose steer is past the lock. Of the 9090 targets the shipped cases can reach (25° to 45°, the path loop's 5.5 m/s to

10.5 m/s and the schedule's friction from 0.58 to 0.92), only three corner targets at 45° change, their steer a hair past the lock. No shipped run visits them, so every run reported here is unchanged. The estimator meets only the friction, not a gust or a load transfer, and the gain schedule assumes the declared mass, inertia and stiffnesses.

OPEN CHALLENGE Holding the Slide Through Slow Actuators

Path-following drift control was first validated on a purpose-built electric research car with near-instant torque and independently driven rear wheels. In simulation, added actuator latency set the sideslip oscillating, and above 95 ms the oscillations became catastrophic. A predictor that runs the model ahead over the mean latency let a production sports car hold circles and figure-eights despite actuator delays over 250 ms. It still lacks a formal proof of closed-loop stability, it was tested only on circles and figure-eights, and tire model fidelity and friction estimation still limit it [18]. Progress would be a delay-tolerant drift law with a proof, shown on arbitrary paths. The chapter's throttle lag and steer delay let a reader meet the same problem.

10.9 Conclusion

A drifting car is held at an equilibrium it leaves by itself, through a tire that has no lateral force left to give, and this simulator lets a learner see both facts. Started on the 35° drift at 8 m/s with its inputs held, the car is 5° off in 2.98 s, at the rate of the slide's root of 1.95/s, and the speed adds a second root of 0.21/s that an analysis at fixed speed does not see. A grip controller spins it at every request of 10° or more. Two driver-like loops hold it until a speed change sets them fighting, while an LQR on the two-input linearization recovers a 10° kick in 0.92 s without overshoot and, scheduled on a friction estimate, holds the drift across a wet patch that spins the fixed design at 10.06 s. A path loop that trims the speed holds the circle within 0.5 m, each figure-eight reversal takes the rear off its friction circle for an instant, and the same car with front drive has no saturated-rear equilibrium when both axles see the same friction. The tire, the drive and the reversal are idealized. A fuller simulator would add the two rear wheels with a differential, a tire whose friction falls as it slides, a reversal planned as a trajectory between equilibria, and an estimator for the vehicle's mass and stiffness. The drift shows that a saturated actuator still leaves a handle when another input shares its limit.

KEY POINTS

- A drifting car is held at a point it leaves by itself, through rear tires that have no sideways force left to give.
- Started on the 35° drift at 8 m/s with its inputs held, the car is 5° off in 2.98 s, at the rate of the slide's root of 1.95/s.
- A grip controller spins the car at every request of 10° or more, and two driver-like loops hold it until a speed change sets them fighting.
- An LQR on the two-input linearization recovers a 10° kick in 0.92 s and, scheduled on a friction estimate, holds the drift across a wet patch.

Where else this applies. Chapter 4 also works a tire at the peak of its force curve, and Chapter 2 holds a reactor at a steady state it leaves by itself.

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10.10 Exercises

10.1 ○ Concept check · 2 min

In the drift of case 3 the rear tire slips at −42.4°, far past its sliding angle of 5.48°, so its force sits on its friction circle. How does the drive torque still move the rear's lateral force?

(a) More torque moves load onto the rear axle, so its friction circle grows and its lateral force grows with it

(b) More torque raises the rear slip angle, and the brush curve's lateral force keeps rising past the sliding angle

(c) The resultant stays on the circle, so more drive force leaves less lateral force and the rear swings further out

(d) It does not: torque changes only the speed, and the lateral force answers only to the steer through the front

10.2 ○ Concept check · 2 min

In case 2 the grip controller, asked for 30° of sideslip, spins the car at 3.61 s, and it spins at every request of 10° or more. What is the reason?

(a) Its integral terms wind up against the steer lock before the sideslip builds, and the stored error throws the car past 70°

(b) Its 10 ms control period is too slow for the slide's root of 1.95/s, so any law at that period loses the car

(c) Its speed loop lifts the throttle as the car slows in the turn, so the rear grips again and snaps the nose around

(d) It steers into the turn for more yaw rate, which assumes a rear force that rises with slip, and a sliding rear cannot check the yaw

10.3 Concept check · 1 min

In case 9 the same car drives its front axle, and on equal friction the solver finds no drift at any target from 5 to 12 m/s and 5° to 45°. Why?

(a) With the rear saturated and undriven, the yaw balance takes the front's whole friction circle for lateral force and leaves none to drive

(b) The front's cornering stiffness of 120 kN/rad is too low for any countersteer to balance the yaw moment of the sliding rear

(c) The LQR's gain schedule has no nodes for front drive, so the equilibrium solver is never called with the front axle driven

(d) The front's drive force pulls the nose toward the velocity, which limits any front-drive drift to the rear's sliding angle

10.4 Theory and calculation · 6 min

At the design point the car drifts on the dry pad ($\mu = 0.9$) at $V = 8\,\mathrm{m/s}$ with $|\beta| = 35°$ and $r^* = 0.927\,\mathrm{rad/s}$. Use $m = 1450\,\mathrm{kg}$, $a = 1.0\,\mathrm{m}$, $b = 1.5\,\mathrm{m}$ and $g = 9.81\,\mathrm{m/s^2}$.

(a) Find the static axle loads and the radius μF_{zr} of the rear friction circle. *Checkpoint:* 8.53 kN, 5.69 kN and 5.12 kN.

(b) In magnitude, the lateral and yaw balances of Equation (10.6) read $F_{yf}\cos\delta + F_{yr} = mrU$ and $aF_{yf}\cos\delta = bF_{yr}$. Find U, the rear lateral force and $F_{yf}\cos\delta$. *Checkpoint:* 6.55 m/s, 3.52 kN and 5.29 kN.

(c) Put the rear force on its circle and find the drive force F_{xr}^*. Then find the drive power from $T^* = 1152\,\mathrm{N\,m}$ and $\omega^* = 38.16\,\mathrm{rad/s}$. *Checkpoint:* 3.72 kN and 44.0 kW.

10.5 Theory and calculation · 6 min

The linearization at the design point has the unstable roots $p_1 = 1.955/\mathrm{s}$ and $p_2 = 0.206/\mathrm{s}$.

(a) Find the doubling time of each mode, and the time the slide's mode takes to grow an error from 0.5° to 5°. *Checkpoint:* 0.355 s and 3.36 s, then 1.18 s.

(b) In case 3 the sideslip error passes 0.5° at 1.82 s and 5° at 2.98 s. Find the growth rate these give, and say which root it matches. *Checkpoint:* 1.98/s, the slide's.

(c) Section 1.4.7 states that feedback on an unstable mode through a delay τ_d is stable for some gain only if $p\tau_d < 1$, and that a loop around an unstable pole needs a bandwidth above about $2p$. Find the longest delay the slide allows, $p_1\tau_d$ at the app's largest steer delay of 200 ms, and the bandwidth floor. *Checkpoint:* 0.51 s, 0.39 and 3.9 rad/s.

10.6 Theory and calculation · 6 min

Equation (10.7) bounds the sideslip of a saturated-rear drift on front drive.

(a) Show that with static loads the front's lateral demand $(b/a)\mu_R F_{zr}$ equals $\mu_R F_{zf}$, and evaluate it on the dry pad. *Checkpoint:* both 7.68 kN.

(b) Evaluate the bound for $\mu_F = 0.9$ on $\mu_R = 0.6$, and for equal friction. *Checkpoint:* 33.9° and 0°.

(c) A designer moves the center of mass to mid-wheelbase, $a = b = 1.25\,\text{m}$, keeping $\mu_F = 0.9$ and $\mu_R = 0.6$. Find the largest saturated-rear drift, and say whether a front-drive drift at 35° is possible on either car. *Checkpoint:* 29.2°; no.

10.7 App challenge · 8 min

Open case 3, "The equilibrium, with no controller", at the default seed and 1× speed. It starts in Manual with Steer and Throttle at the drift's values to the stations' resolution.

(a) Before pressing Run, read δ^*, r^* and F_x^* on the Circle view, and the unstable roots on the Loop view. *Checkpoint:* −25.0°, 0.93 rad/s, 3.72 kN; +1.95 and +0.21/s.

(b) Press Run and let the app pause itself. Read the Run tile. From the exported CSV column `beta_deg`, find the first rows at which $|\beta|$ differs from 35° by more than 0.5° and by more than 5°, and the growth rate the two times give. *Checkpoint:* SPUN at 4.29 s; 1.84 s and 3.00 s (within 0.02 s); about 1.98/s.

(c) Set Target sideslip to 25°, press Put the car on its drift equilibrium, press Reset and Run. When is the sideslip 5° from 25°? *Checkpoint:* about 3.16 s.

10.8 App challenge · 7 min

Open case 5, "LQR about the equilibrium, and a kick", at the default seed and 1× speed. A 10° kick arrives at 8 s.

(a) Run the 20 s as shipped. Read the In the band and Cones hit tiles. From the CSV column `beta_deg` after 8 s, find the largest $|\beta|$ and the time after the kick from which $|\beta|$ stays within 2° of 35°. *Checkpoint:* 19.7 s, 0; 35.0°; 0.92 s (within 0.05 s).

(b) Press Reset, set Controller to Nested, and run the 20 s again. Read the same quantities. *Checkpoint:* 18.5 s, 4; 42.1°; 3.76 s (within 0.05 s).

(c) Explain why the LQR recovers sooner and without overshoot. *Checkpoint:* its gain moves both actuators at once.

10.9 App challenge · 8 min

Open the challenge, "Donut and figure-eight", at the default seed and 5× speed. Before each run after the first, press Reset and change only the setting named, keeping the earlier changes. Run until the app pauses itself or 60 s pass.

(a) Run it as shipped. Read the Run, Cones hit and Score tiles. *Checkpoint:* SPUN at 32.54 s, 6 cones, score 0.

(b) Set Controller to LQR. The schedule already reads Estimated $\hat{\mu}$. Read the clock when the app pauses, and the Run, Cones hit and Transitions tiles. *Checkpoint:* about 46.98 s, OFF PAD, 7 cones, 2 of 2.

(c) Set Path loop to On, and read the Score tile and the Challenge target. *Checkpoint:* 58 (56.2 s and 2 transitions), 0 cones, met.

(d) Set Friction in the schedule to Dry pad, and read the Run tile. Which weakness of the shipped settings did each of the three changes address? *Checkpoint:* SPUN at 37.41 s.

10.10 Application area · 3 min

A team wants to start the tuning of a real rear-drive drift car with an open differential from the simulator's scheduled LQR.

(a) Name two assumptions of the model that bear most on whether the equilibrium carries over to that car, and say which way each would move it. *Checkpoint:* single track with no lateral load transfer or differential; sliding friction that does not fall below the peak.

(b) Which quantity in the simulator would you compare against a first test on the real car before trusting the gain? *Checkpoint:* the equilibrium: steer, yaw rate and drive at a steady drift.

10.11 Application area · 3 min

The friction estimate of Equation (10.11) updates only while the rear's slip speed exceeds 1 m/s.

(a) Why does it hold its value while the rear grips, and what does that mean for a road car whose stability control keeps every tire gripping? *Checkpoint:* on grip the force is inside the circle, so the estimate cannot learn μ.

(b) At the wet patch's edges the estimate is up to 0.30 off for a moment. What does a schedule on $\hat{\mu}$ do during that moment? *Checkpoint:* it uses the wrong equilibrium and gain until the estimate catches up.

10.12 Review the agent · 5 min

An AI assistant reviews the LQR of case 6 for a test track whose wet quarter has $\mu = 0.6$. It writes: "The LQR keeps at least 60° of phase margin and tolerates a gain reduction to one half. A friction change from 0.9 to 0.6 is well inside those margins, so the dry pad's design will hold the drift on the wet quarter, and scheduling on $\hat{\mu}$ adds complexity for no benefit." Find the flaw, and say how you would check the claim on the simulator.