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CHAPTER 9

Pension Funding Ratio Control Through a Year-Late Valuation and a Contribution Cap

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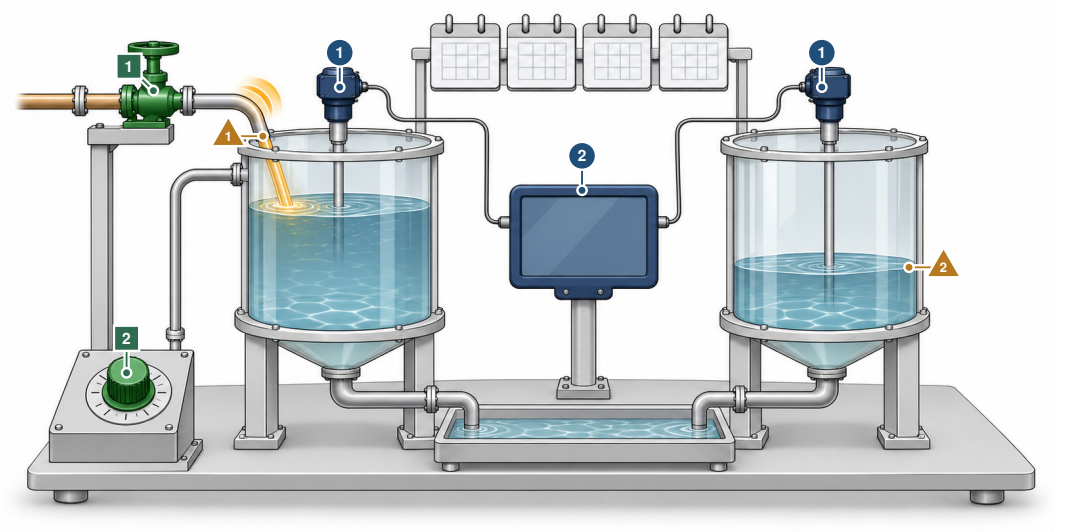
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CHAPTER 9

Pension Funding Ratio Control Through a Year-Late Valuation and a Contribution Cap



The controller moves 1 the sponsor's contribution 2 the share held in equities

The controller reads 1 assets and liabilities, counted once a year 2 the published valuation, a year late

What disturbs it 1 market returns, a crash among them 2 a fall in the discount rate, which raises the liabilities

The left tank holds the assets and the right tank the liabilities, both counted once a year for a display that shows them a year late, while the contribution valve refills the assets far more slowly than a crash can drain them. Illustration, not to scale.

IN PLAIN TERMS **Refilling a Fund From Last Year's Count**

The main idea. A pension fund holds money today for pensions it must pay for decades. Once a year the employer decides how much to pay in, using a count of the fund's money and promises that is already a year old.

The challenge. A market crash can drain the fund far faster than the largest payment the employer can afford will refill it. And because the count is late, each payment answers last year's fund, not this year's.

Why it matters. The payment rule decides who carries the risk: the employer, through payments that jump, or the fund, through deeper shortfalls.

THREE WAYS TO READ THIS CHAPTER

Explore	11 min	the first page, the two history segments (Section 9.2), the fact on missing filings (Section 9.3), the fact on the late valuation and the two try-its (Section 9.6) and the key points
Study	45 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	15 min	the first page, the results (Section 9.6), Technology in Practice (Section 9.7), Limitations (Section 9.8) and the open challenge

The control question A sponsor resets a pension contribution once a year from a valuation a year old, through a cap that refills the fund far more slowly than a market return can drain it. What can a contribution rule achieve under that delay and that cap, and what does a recovery deadline add?

Learning objectives

- Derive the ratio's open-loop growth and the recovery time that a contribution cap sets after a crash.
- Compute the poles of an annual amortization rule on a year-late valuation, and the spread period below which it overshoots.
- Predict the contributions that the calendar's quantum and move limit let a rule apply.
- Explain what a projected estimate sees of a crash, and what a funding buffer buys a predictive plan.
- Judge a funding rule on a declared seed set, not on one path.

Prerequisites Sections 1.4.2, 1.4.5, 1.4.7, 1.6.2, 1.6.3 and 1.6.4. Volume 2, Chapter 10 uses the same calendar on a budget.

The app https://apmonitor.com/apps/pension-funding/

See also Volume 4, Chapter 8 and Volume 4, Chapter 14, stocks with a floor drained faster than they are refilled. Volume 9, Chapter 7, a margin held against an untrusted forecast, and Volume 9, Chapter 2, predictive control behind a dead time.

9.1 Introduction

A defined-benefit pension fund promises benefits that will be paid for decades and holds assets against their present value, so the ratio of assets to liabilities is the level its sponsor, its actuary and its regulator all watch. In the simulator of this chapter, a one-year equity crash of 30 % together with a fall of 1.5 points in the discount rate takes a fully funded fund to a ratio of 0.648 within the year, while a sponsor capped at 12.5 % of payroll can pay at most 2.5 points above the normal cost, which raises the funding ratio by at most 0.00625 a year. The seven-year recovery rule, started by the valuation of year 5, asks for a ratio of 0.8 by year 12, the valuation of year 12 reads 0.698 while the fund stands at 0.707, and the fund regains 0.8 only at 20.23 yr. With the cap at 30 % of payroll the same rule meets the deadline, with contributions that climb to 25.4 %.

Actuaries have treated the setting of contributions as feedback for at least three decades. The classical funding methods add to the normal cost a constant fraction of the unfunded liability [1], and the adjustment was described as a negative feedback component in which the current status is compared with a target and corrective action is taken [2]. Stochastic models then posed the fund level as the controlled variable and the contribution rate as the controlling one, with weighted penalties on the deviations of both [3, 4], and continuous-time models optimized the contribution and the asset allocation together [5, 6]. Treating the fund in a control text therefore needs no translation, because its own literature already speaks of targets, disturbances, delays and quadratic costs.

The structure is general. A stock whose deviation grows on its own, measured once a year and a year late, refilled through a bounded, rate-limited and quantized valve whose authority is small next to the disturbance, describes an endowment drawn down by a spending rule, an insurer's capital, a public debt when interest exceeds growth, a startup's cash, and a storage tank traded against uncertain prices. Volume 2, Chapter 10 treats such a debt with an annual budget, and Volume 4, Chapter 8 such a cash tank with a floor. Each of its parts (dead time, a sampled measurement, an unstable open-loop pole and actuator limits) is standard in process control [7], and Sections 1.4.2 and 1.4.5 treat the first two. The belief the simulator corrects is that a deadline or a faster rule restores a fund. In its crash case every amortization period from 3 to 30 years misses the deadline and ends near a ratio of 0.738, because every one of them sits on the sponsor's cap. A single path also cannot show the trade the rules are chosen by. On seed 11 a thirty-year amortization period ends better funded than a seven-year one, 0.831 against 0.809, on seed 101 it ends at 0.630 against 1.259, and over five declared seeds its lowest published ratio averages 0.598 against 0.695.

However, the actuarial results are stated for independent returns, linear rules without bounds and limiting variances [1, 2, 8], and the recovery rules are written in law with thresholds, deadlines and benefit reductions [9, 10], so a student seldom has one fund on which a rule, its bounds, its calendar and its deadline act together on the same seeded returns. This chapter works with one fund of this kind in full and measures on it what the valuation delay, the sponsor's cap and the return draws do to six ways of setting the contribution. The actuarial models, the amortization rules and the optimizer are established methods from the works cited below. The simulated fund keeps asset and liability ledgers that reconcile at every step, with an annual valuation, a projected estimate, the decision calendar that the economics apps share and a quadratic program, all written out below. Nine cases, a challenge and a reference each isolate one mechanism.

Section 9.2 traces the funding problem from actuarial spreads to recovery plans. Section 9.3 writes the fund as two stocks, with its return, its revaluation, its year-late valuation and its recovery clock. Section 9.4 gives its open-loop and closed-loop character, including the discrete-time poles of the amortization rule with a year of delay and the bound that the cap sets after a crash. Section 9.5 sets out the rules on their calendar, from amortization to the predictive plan. Section 9.6 reports the measured runs, on the five seeds the application declares and on a hundred more. Section 9.7 describes the real valuation, contribution and calendar, Section 9.8 states the limitations and Section 9.9 concludes. The exercises in Section 9.10 use the cases.

9.2 Funding Rules From Actuarial Spreads to Recovery Plans

The arithmetic of a funded pension scheme predates its control theory. The simple model that later stochastic work extends was proposed by Trowbridge in 1952, with an equation of equilibrium that ties the actuarial liability, the normal cost and the benefit outgo together when every assumption is borne out by experience [2]. A funding method fixes the contribution at each valuation as the normal cost plus an adjustment that liquidates the unfunded liability, the difference between the actuarial liability and the fund. In 1988 a simple model with random rates of return was used to study the variability of contributions and fund levels under methods whose adjustment is a constant fraction k of that unfunded liability [1]. Its limiting variances showed a range of spread periods in which a longer period lowers the variance of the contribution and raises that of the fund, and beyond which both rise [2].

Delay entered next, in the work that the history segment below describes [11]. The same group then wrote the problem in the language of optimal control. The 1994 model treated the fund as the controlled variable and the contribution as the controlling variable with weighted penalties on both deviations, the contribution rate risk and the solvency risk [3, 4]. Three ways of adjusting the normal cost as gains and losses arise were compared under random returns [8], and amortizing each gain or loss directly was later found to give more variable funding and contributions than spreading them in proportion, the spread being the contribution control that minimizes mean square deviations [12].

> ⧖ HISTORY **1992 to 2002: The Year-Late Valuation Enters the Theory**
>
> An actuarial valuation reports a fund level only after data have been collected and processed and the accounts prepared. A model of 1992 therefore let the adjustment to the contribution depend on the unfunded liability of the year before, and a 1995 analysis compared that one-year delay with no delay at all [2]. With the delay, both limiting variances, of the fund and of the contribution, exceed their values without it, because the information about the fund since the last year has been lost. A trade between the two variances remains, and so does an optimal range of spread periods. The same analysis noted that spread periods of 20 to 25 years were common practice in the United Kingdom [2]. A study of 2002 then controlled the fund ratio rather than the fund level, with a one-year delay in the accounts, and derived its funding policy through the separation principle [4].

Continuous time added the asset allocation as a second handle. A model with several risky assets and random benefit outgo found that a quadratic loss leads to investing more in risky assets when the funding level is low and shifting toward low-risk assets as it rises, which its author called a counterintuitive strategy, and it listed the constraints that complicate the analysis, among them a sponsor's limit on the contribution rate such as twice the normal rate [5]. Minimizing the contribution rate risk together with the solvency risk was shown to lead in several cases to a spread method of funding [6]. For defined-contribution plans, deterministic lifestyling moves gradually from equities to bonds by preset rules [13], which is the family to which the application's glide path belongs, keyed here to the funding ratio rather than to age.

> ⧖ HISTORY **2006: A Seven-Year Rule on Last Year's Ratio**
>
> The Pension Protection Act, approved in the United States on August 17, 2006, rewrote the minimum funding rules of single-employer defined-benefit plans. An underfunded plan's minimum required contribution became its target normal cost plus a shortfall amortization charge, which repays each year's shortfall base in level annual installments over seven plan years. A plan was placed in at-risk status when its funding target attainment percentage for the preceding plan year was below 80 percent and, computed with additional actuarial assumptions, below 70 percent. For plan years beginning in 2008 the act let that preceding-year percentage be determined by methods of estimation that the Secretary of the Treasury might provide [9]. The test therefore reads a ratio a year old, as the valuation of this chapter does.

The law fixed the period and the deadline, in the United States through the act of 2006 that the second history segment describes [9]. In the Netherlands the solvency crisis of 2001 to 2004 moved most pension funds to average-wage plans with solvency-contingent indexation [14]. The Dutch Pension Act as it read in the first half of 2015 required a fund below its required own funds to submit a recovery plan reaching them within at most ten years, and measures that may reduce accrued rights when its policy funding ratio stays below the minimum at five consecutive annual determinations [10]. Two facts outside the rules set the stakes. The size of the promise depends on the discount rate, so that state pension liabilities valued at zero-coupon Treasury yields came to $4.43 trillion against $3.20 trillion at rates matched to state debt [15], and the cash a sponsor pays is not free, because capital expenditures fall with mandatory contributions, most visibly at firms facing financing constraints [16].

9.3 A Fund of Two Stocks

The simulated fund is one open defined-benefit plan with a stationary membership, drawn in Figure 9.1 as the tanks and valves it is. Money is measured in units of the initial liability L_0, time in years, and every flow is a rate per year. Table 9.1 classifies the signals. Every coefficient is a teaching value of Table 9.2, chosen for an imaginary plan and not estimated for any fund.

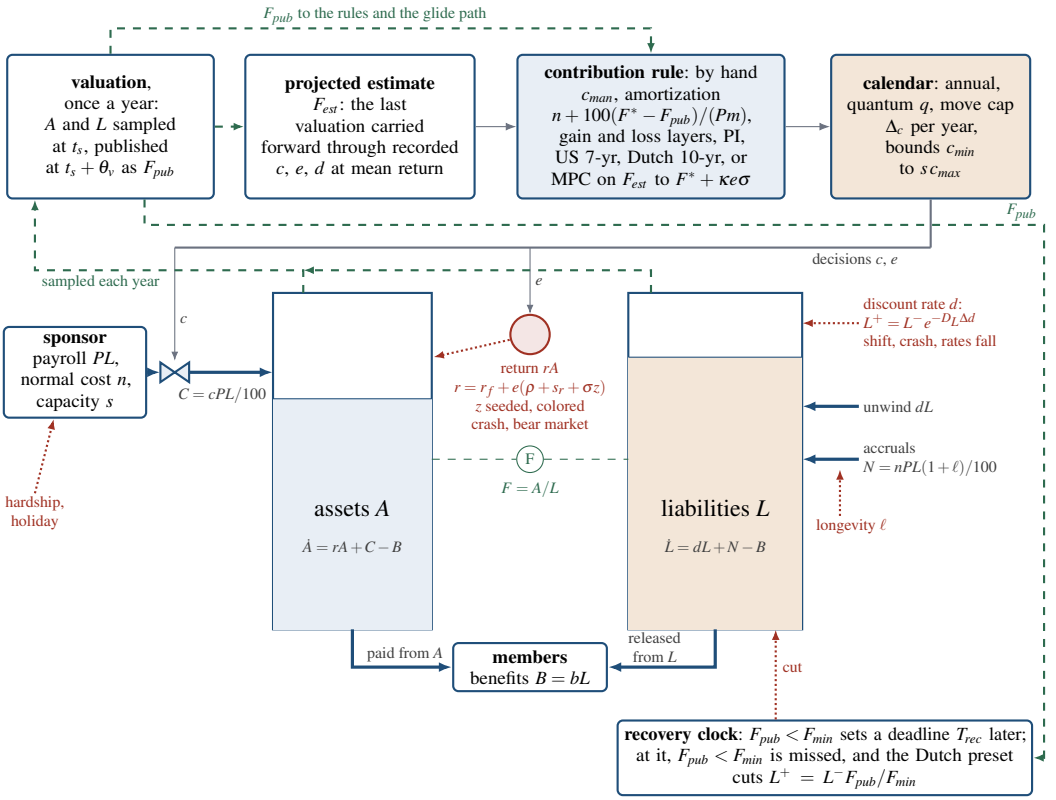


Figure 9.1 The defined-benefit fund as two stocks, not to scale. Contributions C from the sponsor and the return rA fill the assets A, the unwind dL and the accruals N fill the liabilities L, and the benefits B leave both. A move of the discount rate d revalues L through its duration D_L. Dotted red arrows are disturbances, thin solid gray arrows the decisions c and e on the calendar, dashed green lines the measurement path, and thick solid blue arrows the flows of money. The valuation samples A and L once a year and publishes F_{pub} a delay θ_v later, the estimate carries it forward to F_{est}, and the recovery clock may cut L at a missed deadline.

Table 9.1 Signals of the Fund by Role, With Their Defaults

Symbol	Role	Default	Range
F	CV, current ratio A/L, shown, not used by rules	1	—
F_{pub}	measured, last published valuation	1	yearly
F_{est}	estimate projected from F_{pub}	1	—
F^*, F_{min}	SP, funding target and recovery threshold	1, 0.8	0.8–1.4, 0.6–1
c	MV, contribution, % of payroll	10	0–30
e	MV, equity share, by hand or glide	0.6	0–1
s_r	DV, equity return shift and scripted shocks	0	±0.2/yr
Δd	DV, discount-rate shift and scripted moves	0	±0.03/yr
ℓ	DV, additional accrual fraction	0	0–0.2
s	DV, sponsor capacity factor	1	0.2–1
z	DV, seeded colored return noise	seed 11	1–9999
A, L	states, assets and liabilities	1, 1	L_0
—	states, seven ledgers of every flow	0	L_0, %·yr
σ_c	output, time-weighted contribution SD	0	points

9.3.1 Assets, Liabilities and the Ratio

Contributions and returns fill the assets, benefits drain them, and the liabilities grow by the unwinding of the discount and by new accruals while the same benefits leave them, as shown in Equation (9.1).

$$\begin{aligned} \dot{A} &= rA + \frac{c}{100} PL - bL \\ \dot{L} &= dL + \frac{n}{100} PL(1+\ell)\lambda - bL, \qquad F = \frac{A}{L} \end{aligned} \tag{9.1}$$

where $P = 0.25$ is the annual payroll as a fraction of the liabilities, c and $n = 10$ are the contribution and the normal cost in percent of payroll, $b = 0.05$ is the benefit outgo as a fraction of the liabilities a year, d is the discount rate, ℓ the additional accrual fraction and λ the product of 1.08 for each longevity event, one otherwise. Payroll, accruals and benefits are all proportional to the liability, so the fund has no cohorts. Each flow is also integrated into a ledger of its own, and the assets and the liabilities are reconciled against their ledgers at every step. Over all eleven cases the larger residual is $2.6 \times 10^{-14} L_0$, revaluations and benefit cuts included.

9.3.2 Return, Discount Rate and Revaluation

The fund earns the risk-free rate on everything and the equity premium, the noise and any scripted shock on its equity share, as shown in Equation (9.2).

$$\begin{aligned} r &= r_f + e\big(\rho + s_r + \sigma z + j\big) \\ z_{k+1} &= a z_k + \sqrt{1-a^2}\, \nu_k, \qquad a = e^{-\Delta t/\tau_z} \end{aligned} \tag{9.2}$$

where $r_f = 0.016\,\text{yr}$, $\rho = 0.04\,\text{yr}$, $\sigma = 0.16$, ν_k is a standard normal deviate from a mulberry32 generator seeded by the return seed, $\tau_z = 1\,\text{yr}$, and z is held over each weekly step $\Delta t = 1/52$ year. An optional crash year sets $j = \ln 0.75$ for a whole year with probability 0.08, drawn from the return seed at each year boundary, and $j = 0$ otherwise, so a drawn year takes 25 % off equities whatever σ is. A crash event adds $s_r = \ln 0.7$ for one year, so at $e = 0.6$ the assets fall by a factor $0.7^{0.6} = 0.807$ over the year from that term alone. The return is a force of interest, and σ is a scale rather than an annual volatility. Integrated over a year, the noise term's spread is 0.857 times $e\sigma$ in standard deviation, 8.23 % at $e = 0.6$, and consecutive annual integrals are correlated at 0.54, which the application's generator reproduces over 200 000 simulated years against 0.858 and 0.543 for the continuous process. With $r_f + 0.6\rho = d_0 = 0.04\,\text{yr}$ the expected return equals the discount rate, the matched point at which the deterministic fund stays at $F = 1$.

The discount rate is the initial 0.04 yr plus the slider shift and any scripted move, and a change revalues the liabilities through their duration, as shown in Equation (9.3).

$$L^{+} = L^{-} \exp\big(-D_L (d_k - d_{k-1})\big) \tag{9.3}$$

where $D_L = 15\,\text{yr}$ and k counts weekly steps, so a fall of 1.5 points raises L by a factor $e^{0.225} = 1.252$. The revaluation enters its own ledger, separate from accruals. A scripted move may be a step, taken over one week, or a ramp, such as two points over five years. The asset return does not depend on d, so the fund holds no bonds whose price would rise as the rate falls.

9.3.3 The Valuation and the Projected Estimate

The valuation is the framework's sampled analyzer, the instrument of Section 1.4.5 with a sample period of a year. At each integer year $t_k \geq 1$ it publishes the assets and liabilities as they stood θ_v earlier, as shown in Equation (9.4).

$$F_{pub}(t) = \frac{A(t_k - \theta_v)}{L(t_k - \theta_v)}, \qquad t_k \leq t < t_{k+1} \tag{9.4}$$

Table 9.2 Model Constants and Disturbances. All Are Teaching Values, Shown in the Show Model Dialog (M) or Among the Disturbances (D, A for Advanced)

Symbol	Meaning	Value	Range	
F_0	initial funding ratio	1	0.5–1.6	M
r_f, ρ	risk-free return, equity premium	0.016, 0.04/yr	0–0.06, 0–0.1	M
σ, τ_z	noise scale, correlation time	0.16, 1 yr	0–0.3, fixed	M
d_0, D_L	discount rate, liability duration	0.04/yr, 15 yr	0–0.08, 3–25	M
P, n	payroll over L, normal cost	0.25/yr, 10 %	0.1–0.5, 5–20	M
b	benefit outgo over L	0.05/yr	0.02–0.12	M
θ_v	valuation delay	1 yr	0–2	M
—	return seed, crash year j	11, off	1–9999, 0/1	M
s_r, s	return shift, sponsor capacity	0, 1	±0.2, 0.2–1	D
Δd, ℓ	rate shift, extra accrual	0, 0	±0.03, 0–0.2	A
—	crash: $s_r = \ln 0.7$ for 1 yr, $\Delta d = -0.015$	event	—	D
—	rates fall 2 points over 5 yr, longevity $\lambda = 1.08$	event	—	D
—	hardship, s halves for good	event	—	D
Δt	integration step and log interval	1/52 yr	—	—

where $\theta_v = 1$ yr, so the valuation of year 5 describes year 4, and before the first one the published value is the initial state. A held valuation is therefore on average a year and a half old, by Equation (1.8) with $t_d = \theta_v$. A crash that begins at year 3 therefore first appears in the valuation of year 5, because the valuation of year 4 describes the instant before it. The projected estimate $F_{est} = \hat{A}/\hat{L}$ restarts from the published A and L at each valuation and integrates Equation (9.1) forward through every recorded step since the sample, by the midpoint rule, with the recorded contribution, equity share, discount rate, accrual and revaluation, at the declared mean return, $s_r = z = j = 0$. It never reads a realized equity return, random or scripted, so a crash reaches it only through the next valuation. Restarting from the instant the valuation describes and predicting forward through the recorded inputs is the second treatment of a late measurement in Section 1.4.7, with no filter gain, and the projection is a predictor across the delay in the sense of Section 1.4.3.

> **+ DID YOU KNOW? Three Years Without a Reading**
>
> The United States insurer of private defined-benefit plans put the underfunding of single-employer plans at $482 billion in 2021. Its report on fiscal year 2024 showed no estimate for 2022 to 2024, because the plans' annual filings were not yet available, and its model projected about $41 billion by the end of 2024 [17].

9.3.4 Recovery Clock and Benefit Cut

A published ratio below the threshold starts a deadline, and a missed deadline under the Dutch preset reduces the liabilities, as shown in Equation (9.5).

$$\begin{aligned} F_{pub} < F_{min} &\Rightarrow t_{dl} = t_k + T_{rec} \\ F_{pub} \geq F_{min} &\Rightarrow \text{deadline cleared} \\ t_k \geq t_{dl},\ F_{pub} < F_{min} &\Rightarrow \text{missed},\ L^+ = L^- F_{pub}/F_{min} \end{aligned} \tag{9.5}$$

where T_{rec} is seven years by default and ten under the Dutch preset. The deadline is judged by the first valuation at or after it. The Dutch cut, made once, brings that valuation to the threshold, applies to the liabilities as they stand, restates the published ratio and pauses the run. Negative assets would be written off as insolvency, which no shipped case reaches.

9.3.5 Initial State, Step and Checks

Every case starts with $A = F_0$ and $L = 1$, its noise at zero and its valuation equal to the initial state, with the first sample due at year 1. Classical fourth-order Runge-Kutta integrates the nine states on the weekly step, with the contribution, the equity share and the noise held over each step. The

revaluation, the estimate, the valuation, the recovery clock with its cut and the next noise value are applied between steps, in that order, and the historian logs every week. The controller runs every fifth step, $5/52 = 0.0962$ yr, the period the application declares, because a nominal month is not a whole number of weekly steps. Section 1.6.1 describes the engine's clocks, its seed and its Reset. In the matched reference run for thirty years the ratio stays at 1 while assets and liabilities both grow to $e^{0.45} = 1.568\,L_0$. Halving the step in the deterministic crash at a fixed 12.5 % moves the ratio at year 15 from 0.776253 to 0.776240, and a reset with the same seed reproduces a case bit for bit. The application passes its 35 checks, 18 of them its own.

9.3.6 Assumptions

An aggregate fund. Payroll, accruals and benefits are fixed fractions of the liabilities, with no cohorts, mortality tables, salary scales, indexation or taxes. A real valuation projects each member's benefits and would move the normal cost with the membership.

A return independent of the discount rate. The assets carry no duration, so a fall in rates revalues only the liabilities. A fund that hedges with long bonds would see part of the revaluation offset on the asset side.

Colored noise. Annual returns correlated at 0.54 make five-year variance larger than independent years would, a standard deviation of 2.82 times $e\sigma$ against $0.857\sqrt{5} = 1.92$. The actuarial results of Section 9.2 assume independent returns.

Simplified rules. The legal presets carry one threshold, one deadline and one cut, not the statutes' many tests.

Teaching values. Every parameter, threshold, deadline and challenge target is a teaching value. Real ones come from a plan's actuarial valuation, the statute that governs it [9, 10] and its regulator, and none of the numbers here describes any fund.

9.4 The Ratio, Its Gain and Its Lag

9.4.1 A Deficit That Grows by Itself

Dividing Equation (9.1) by L gives the funding ratio as one linear equation in the ratio itself, as shown in Equation (9.6).

$$\dot{F} = (r - g_L)F + \frac{P}{100}c - b, \qquad g_L = d + \frac{n}{100}P(1+\ell)\lambda - b \tag{9.6}$$

where g_L is the growth rate of the liabilities, 0.015 yr at the defaults. At the matched point $r - g_L = 0.025$ yr, so a deficit held at constant contributions grows on its own and doubles in 27.7 yr. The benefits are a fixed fraction of L but come out of A, and a fund below full funding pays a larger share of its own assets. A point of payroll moves $\dot{F}$ by only $P/100 = 0.0025$ a year. After the crash case's fall of 1.5 points in d the open-loop gain $r - g_L$ becomes 0.040 yr, and at the 12.5 % cap Equation (9.6) has an unstable equilibrium at $F = 0.469$. From its low of 0.648 the ratio therefore climbs at only 0.0072 yr, and at the 30 % cap at 0.051 yr. The engine confirms the bound, with the seven-year rule reaching 0.8 at 20.23 yr under the low cap and at 10.60 yr under the high one.

9.4.2 Amortization as a Proportional Gain

The amortization rule sets the contribution to the normal cost plus the deficit spread over m years, $c = n + 100(F^* - F)/(Pm)$, which on Equation (9.6) at the matched point gives a closed-loop pole of $r - g_L - 1/m = 0.025 - 1/m$ per year. The application's own check finds -0.041667 at $m = 15$ against the same value analytically, to 1.6×10^{-13}. The pole is -0.175 at $m = 5$ and -0.0083 at $m = 30$, and it reaches zero at $m = 40$, beyond the slider's range of 3 to 30, so a long enough spread no longer brings a deficit back at all.

The application decides once a year on a valuation one year old. With the contribution held for exactly a year (the application's decisions fall 0.9615 or 1.0577 years apart, a year apart on average)

Table 9.3 Contribution and Allocation Settings. A Marks an Advanced Control

Symbol	Setting	Default	Range
—	contribution rule	amortization	6 rules
—	decision calendar	annual	quarterly, every run
c_{man}	request in MANUAL, % payroll	10	0–30
m	amortization period	15 yr	3–30
c_{min}, c_{max}	contribution bounds (A for c_{min})	5, 30	0–15, 10–30
Δ_c, q	move limit per year, quantum (A)	3, 0.1 points	0.5–10, 0.01–0.5
T_{rec}	recovery deadline (A)	7 yr	3–15
—	holiday above $F_{pub} = 1.2$ (A)	off	on
e	equity share	0.6	0–1
e_{hi}, e_{lo}	glide endpoints (A)	0.6, 0.3	0–1
F_1, F_2	glide band (A)	1.0, 1.3	0.6–1.2, 1–1.6
τ_I, τ_D	PI integral and derivative times (A)	15, 0 yr	0–30, 0–5
H	MPC horizon (A)	15 yr	5–20
κ	funding buffer, in $e\sigma$	2	0–5
w_F, w_c	funding weight (A), smoothing weight	500, 1	10–2000, 0.1–20

and set from the deficit $x = F - 1$ of the year before, the deviation obeys Equation (9.7).

$$x_{k+1} = \phi x_k - \frac{\gamma}{m} x_{k-1}, \qquad \phi = e^{0.025}, \quad \gamma = \frac{e^{0.025} - 1}{0.025} \tag{9.7}$$

whose characteristic roots solve $z^2 - \phi z + \gamma/m = 0$, the equation of a correction that acts on information two decisions old, Equation (1.9), with the fund's growth factor ϕ beside the first term and γ/m in place of g. Without the delay the single root is $\phi - \gamma/m$. At $m = 5$ the delayed loop has roots 0.758 and 0.267 against 0.823 without the delay, at $m = 15$ 0.955 and 0.071 against 0.958, and the roots become complex only for $m < 3.85$ years, $0.513 \pm 0.273\,j$ at $m = 3$. The delay therefore does not slow a deterministic recovery. A rule that acts on last year's larger deficit pays more while the deficit shrinks, which raises the variability of the contribution, as Section 9.6 measures, in the direction the stochastic analysis found for the limiting variances [2]. Equation (9.7) ignores the move cap and the quantum, which bind at the first decision of case 4, where the rule asks for 18.0 points and the cap allows 13.0.

9.5 Deciding Once a Year

Every contribution, whether requested by hand, by a rule or by the optimizer, passes the same calendar and the same shaping before it reaches the valve. Table 9.3 lists the settings.

9.5.1 Annual Decisions, Their Grid and Their Cap

The calendar and shaping functions are those of `shared/src/framework/core/cadence.js`, which the central bank of Volume 2, Chapter 11 also decides through. The annual calendar has an instant at every integer year, the quarterly one at every quarter, and the continuous one makes every controller execution a decision. A decision waits for the first controller execution not earlier than its instant, so on the five-week execution grid the annual decisions fall at years 0, 1.0577, 2.0192, 3.0769, 4.0385 and 5.0000, a pattern that repeats every five years, and the quarterly ones 0.1923 or 0.2885 years apart. At a decision the request c^w is shaped as shown in Equation (9.8).

$$\begin{aligned} \tilde{c} &= q\,\mathrm{round}(c^w/q), \quad \Delta_q = q\lfloor(\Delta_c\tau + \chi_k)/q\rfloor \\ c_k &= \mathrm{clip}_{[c_{lo},c_{hi}]}\big(c_{k-1} + \mathrm{sat}_{\Delta_q}(\tilde{c} - c_{k-1})\big) \\ \chi_{k+1} &= \Delta_c\tau + \chi_k - |c_k - c_{k-1}| \text{ if in } [0,q), \text{ else } 0 \\ c_{hi} &= \max(0, s\,c_{max}), \qquad c_{lo} = \min(c_{min}, c_{hi}) \end{aligned} \tag{9.8}$$

where ties round away from zero, the bounds snap to the grid, τ is the time since the last decision, and the credit χ carries the part of the allowance $\Delta_c\tau$, 2.88 or 3.17 points as τ is 0.9615 yr or 1.0577 yr,

that flooring to whole quanta left unused, so every calendar ramps at Δ_c. At the decision made while a case is set up, τ is a whole calendar step. The contribution then holds until the next decision. A scripted holiday, or the holiday switch while $F_{pub} > 1.2$, sets $c = 0$ outside this shaping, and a cap that falls below the contribution in force cuts it at once through the clip. The equity share, by hand or from the glide path, is set at the same decisions. In MANUAL the learner's request passes Equation (9.8) exactly as a rule's does.

9.5.2 Amortization, the Legal Presets and the Holiday

The amortization rule is the proportional law of Section 9.4 on the published ratio, as shown in Equation (9.9).

$$c^w = n + \frac{100(F^* - F_{pub})}{Pm} \tag{9.9}$$

The gain $100/(Pm)$ is in points of payroll per unit of funding ratio, 26.7 at $m = 15$, and surpluses lower the contribution exactly as deficits raise it, down to c_{lo}. The historical United States preset is Equation (9.9) with $m = 7$ and a seven-year deadline, and the Dutch-style preset uses $m = 10$ with a ten-year deadline and the cut of Equation (9.5). Both are teaching simplifications. The 2006 act amortizes each year's shortfall base in its own seven-year layer and reduces the normal cost by a surplus rather than spreading it [9], which the application's Model dialog and case 5's card also state, and the Dutch text measures its funds against required and minimum own funds rather than a ratio of 0.8 [10].

9.5.3 Gain and Loss Layers

The gains-and-losses rule recognizes each valuation's new deficit apart from the deficit already being repaid, and amortizes it in a layer of its own, as shown in Equation (9.10).

$$\begin{aligned} G_j &= \frac{100(F^* - F_{pub,j})}{P}, \qquad \delta_j = G_j - (G_{j-1} - \Pi_{j-1}) \\ \Pi &= \sum_{t_j > t-m} \frac{\delta_j}{m}, \qquad c^w = n + \Pi \end{aligned} \tag{9.10}$$

where G_j is the deficit at valuation j in points of payroll times years, Π_{j-1} is the payment scheduled before it, and each layer lasts m years. A loss is therefore repaid within m years of the valuation that first reports it, rather than refinanced over a fresh m years at each valuation as in Equation (9.9). This is the structure of the act's shortfall bases [9] and of the direct amortization of gains and losses that the actuarial literature compared with spreading [8, 12].

9.5.4 Proportional-Integral Control

The PI rule is the framework's PID in the velocity form of Section 1.6.2, executed only at decisions on the published ratio, as shown in Equation (9.11).

$$\begin{aligned} e_k &= F^* - F_{pub,k}, \qquad S_k = S_{k-1} + K_c(e_k - e_{k-1}) \\ \tilde{u}_k &= S_k + I_{k-1}, \qquad u_k = \mathrm{clip}_{[c_{lo}, c_{hi}]}(\tilde{u}_k) \\ I_k &= I_{k-1} + \left[\frac{K_c}{\tau_I} e_k + \frac{c_k - \tilde{u}_k}{\tau_I} \right] \tau_k \end{aligned} \tag{9.11}$$

where $K_c = 100/(Pm)$, the amortization gain, $\tau_I = 15\,\mathrm{yr}$, τ_k the time since the last decision, and the integral starts at the request in force, ten points, so that the normal cost is carried by I rather than a bias. The symbols map onto Equations (1.15) to (1.18) with S_k the running sum and $\tilde{u}_k$ the output before limiting. As there, only the emitted contribution is clipped and only the integral is back-calculated, here with the tracking time τ_I. A derivative on the measurement, filtered over one year, is available and off as shipped. The output u_k, slewed to the move allowance, is the request c^w

of Equation (9.8), and the integral is back-calculated against the applied c_k, so the move limit, the quantum, a holiday and a falling cap all unwind it. Under every other rule and in MANUAL the PI tracks the contribution in force at each decision, so a switch to it is bumpless.

9.5.5 The Glide Path

With the glide path on, the equity share falls linearly with the published ratio across a band, as shown in Equation (9.12).

$$e = e_{hi} + (e_{lo} - e_{hi})\,\mathrm{clip}_{[0,1]}\Big(\frac{F_{pub} - F_1}{F_2 - F_1}\Big) \tag{9.12}$$

which de-risks as the fund improves, the direction a quadratic loss also chose [5]. At the defaults the share is 0.6 below $F_{pub} = 1.0$ and 0.3 above 1.3, and the expected return falls from 0.040 yr to 0.028 yr, below the discount rate.

9.5.6 A Certainty-Equivalent Recovery Plan

The MPC solves a quadratic program (QP) at each decision on the linear ratio model of Equation (9.6), with the mean return $\bar{r} = r_f + e\rho$ at the declared $r_f = 0.016\,\mathrm{yr}$ and $\rho = 0.04\,\mathrm{yr}$ and the equity share in force, g_L at the discount rate in force, and an exact zero-order hold over the calendar's step T_s (one year, a quarter, or one controller period on the continuous calendar), through the framework's `core/linmpc.js`. From the projected estimate $x_0 = F_{est}$ it chooses the contributions u_j of Equation (9.13).

$$\begin{aligned}
\min_{u,\varepsilon}\ & \sum_{k=1}^{N} w_F (x_k - \bar{F})^2 + 4 w_F (x_N - \bar{F})^2 \\
& \quad + w_c \sum_j (u_j - u_{j-1})^2 + 10^{-3} \sum_k (u_k - n)^2 \\
& \quad + 5000\,\varepsilon^2 + 100\,\varepsilon \\
\text{s.t. } & x_{k+1} = a_d x_k + b_d u_k + c_d \\
& x_k \geq F_{min} + \kappa e \sigma - \varepsilon, \qquad \varepsilon \geq 0 \\
& c_{lo} \leq u_j \leq c_{hi}, \qquad |u_j - u_{j-1}| \leq \Delta_c \tau + \chi
\end{aligned} \tag{9.13}$$

where $\bar{F} = F^* + \kappa e \sigma$ on the declared $\sigma = 0.16$, 1.192 at the defaults, $N = H/T_s = 15$ annual steps of exactly one year, u_0 is the contribution in force, and the input is held over ten move blocks, nine of one year and one of six. The QP has 11 variables and 56 constraint rows. The first move is rounded to the quantum by Equation (9.8), and the plan is drawn in the Recovery plan view. Over the 15 decisions of case 9 the Goldfarb–Idnani dual active-set solver [18] needs a mean of 2.80 iterations and at most 8, with no failure, and the soft floor binds in four of the plans, those of years 8.08 to 11.06. Section 1.6.3 gives the framework's MPC option in general. The buffer $\kappa e \sigma$ is a margin held above a floor against a disturbance the plan does not predict, as the empty flood space of Volume 9, Chapter 7 is held against an inflow its forecast cannot promise.

9.5.7 Views, Status and the Hand Station

The Fund view draws the asset tank beside the liability column with their flows and the return as weather, the Trade-off view the five-seed means of Figure 9.2(b), and the Recovery plan view the realized ratio, the labeled threshold and deadline, and the MPC's last plan from the year it was solved. Badges give the current and published ratios, the contribution, the equity share and the time left to a deadline, and the status line names UNDERFUNDED, RECOVERY PLAN, DEADLINE MISSED, CUT, HOLIDAY, SPONSOR CAP, CONTRIBUTION FLOOR and CRASH as they apply. In MANUAL the learner sets the request, and in either mode the equity share or the glide path. A helper sets the request to the normal cost, which holds the matched reference exactly. A case is

Table 9.4 The Cases, the Challenge and the Reference

#	Change from the defaults, run length	Teaching point
1	MANUAL at 10 %, bear market of −25 % over years 2 to 5, 15 yr	the valve by hand
2	$m = 7$, seed 101, 30 yr, Trade-off view	fast against smooth
3	gain and loss layers, $m = 15$, 30 yr	layers or a spread
4	$\sigma = 0$, $F_0 = 0.9$, $m = 5$, 15 yr	the late valuation
5	$\sigma = 0$, US 7-yr, cap 12.5 %, crash at 3, 15 yr	no cash, no recovery
6	US 7-yr, $e = 0.9$, crash at 3, 15 yr	risk in the recovery
7	glide on, $F_0 = 1.25$, 30 yr	de-risking
8	$F_0 = 1.25$, holiday to 5, crash at 5, 20 yr	the holiday's bill
9	MPC, glide on, crash at 3, 15 yr	a plan on an estimate
ch	US 7-yr, cap 25 %, 20 % at start, $F_0 = 1.25$, crash at 3, rates −1 point over 4 to 8	survive both
ref	$\sigma = 0$, defaults, 15 yr	the matched fund

complete once run for its length with an observation recorded, and the challenge once its verdict is met, under the rules of Section 1.6.4.

9.5.8 Fund, Rules and Plan

Equations (9.1) to (9.5) are the simulated fund. The amortization, layer and PI rules hold no model beyond n and P in their gain. The MPC's model is Equation (9.6) at the declared mean returns and noise scale, with the allocation, the discount rate and the accrual frozen at their present values, started from an estimate that knows the recorded flows and the yields but no realized equity return, random or scripted. A learner's change of r_f, ρ or σ is thus a model mismatch, which the plan meets as it happens. Planning from an estimate with a certainty-equivalent controller is the separation of estimation from control that the 2002 actuarial study derived for a one-year accounting lag [4], with a deterministic projection here in place of its minimum mean-squared-error estimate. The models they stand for are the discrete-time actuarial models with independent annual returns [1, 2, 11], the continuous-time control models [5, 6], and a plan's full actuarial valuation. The application fits nothing to data.

9.6 Fifteen and Thirty Years at a Time

Table 9.4 lists the cases. Each is selected from the dropdown, changed as its row says and run headless on the engine, on its own return seed, 101 for case 2 and 11 for the others, unless another is named. A change made before Run gives the same run as the change followed by Reset, except a disturbance button, whose fired event a Reset clears. Table 9.5 collects the outcomes. Because the plant is stochastic, Table 9.6 repeats the stochastic cases on the five seeds the application declares, 11, 29, 47, 71 and 101, and Table 9.7 the challenge. The scorecard's integrated error grows with the run and is not quoted. The exported CSV file is logged weekly and holds the current, published and projected ratios, the contribution and its request, the equity share, every flow, the contribution SD, the lowest valuation and the times of the last decision and of the last valuation's sample and report.

9.6.1 The Valve by Hand

Case 1 holds the contribution at the normal cost through a bear market that takes 25 % off equities over three years, which at $e = 0.6$ is a factor $0.75^{0.6} = 0.842$ on the assets from that term. On seed 11 the fund peaks at 1.38 at 2.4 yr, falls to 0.945 at 9.19 yr and ends at 1.303, because that seed's draws are kind. On the five declared seeds the same unchanged hand ends between 0.262 and 1.303, three of the five valuations fall below 0.8, and two recovery deadlines are missed. A hand that requests 15 % from the start on seed 11 keeps every valuation at or above 1.000 and ends at 1.554, at a cost of 0.631 rather than 0.421 initial liabilities of contributions.

Table 9.5 Measured Cases on Their Own Seed, 101 for Case 2 and 11 Otherwise, With Final and Lowest Published Ratios, Contribution SD, and Mean and Largest Applied Contribution

Case and configuration	$F(T)$	F_{pub}^{min}	σ_c	$\bar{c}$	c^{max}	note
1, MANUAL at 10 %	1.303	0.954	0.00	10.0	10.0	
1, MANUAL at 15 %	1.554	1.000	0.00	15.0	15.0	
2, $m = 7$	1.259	0.530	8.41	19.3	30.0	missed
2, $m = 30$	0.630	0.386	3.43	13.9	18.2	missed
3, layers, $m = 15$	0.724	0.785	1.92	6.2	11.8	plan
3, amortization	0.797	0.860	2.58	7.2	12.8	
3, PI, $\tau_I = 15$ yr	0.731	0.793	2.04	6.3	11.3	plan
4, $\theta_v = 1$ yr	0.998	0.900	2.35	12.9	17.6	
4, $\theta_v = 0$	0.994	0.900	1.91	12.9	16.6	
4, $\theta_v = 2$ yr	1.002	0.900	2.93	13.0	18.0	
4, $m = 3$, $\theta_v = 1$ yr	1.000	0.900	3.85	12.9	21.0	1.004
5, US 7-yr, cap 12.5 %	0.738	0.648	1.18	11.7	12.5	missed
5, cap 30 %	0.956	0.648	5.63	16.5	25.4	met
5, Dutch 10-yr	0.812	0.648	1.18	11.7	12.5	cut
6, $e = 0.9$	1.499	0.777	3.64	9.2	17.1	
6, $e = 0.3$	0.994	0.710	6.41	14.2	24.2	
7, glide on	0.796	0.858	2.56	6.7	12.8	
7, glide off	1.088	1.186	0.37	5.1	7.0	
8, holiday	1.165	0.879	4.17	6.0	13.2	
9, MPC, $\kappa = 2$	1.331	0.839	2.85	12.9	18.0	
9, MPC, $\kappa = 0$	1.126	0.739	3.52	9.4	14.8	
9, US 7-yr	1.245	0.773	4.40	11.4	19.1	
ch, as shipped	1.379	0.863	3.91	9.3	17.0	not met
ch, suggested	1.158	0.954	1.46	18.9	21.0	met
ref	1.000	1.000	0.00	10.0	10.0	

T, the case's run length. Plan, started at year 30. 1.004, the peak of F.

9.6.2 Spread Period, Layers and the Trade

Case 2 compares spreading the deficit over 7 and over 30 years on the same draws. On seed 11 the longer period halves the contribution standard deviation, 1.86 against 3.74 points, and also ends better funded, 0.831 against 0.809, with a lowest valuation of 0.901 against 0.864. The case therefore ships on seed 101, where the SD falls from 8.41 to 3.43 points while the lowest valuation falls from 0.530 to 0.386 and the final ratio from 1.259 to 0.630, the trade its card describes. Seeds 29 and 71 show the same order, 11 and 47 do not. Over the five seeds the SD averages 2.92 points at $m = 30$ against 6.11 and the lowest valuation 0.598 against 0.695. The application's trade-off view averages the within-path standard deviations over the five seeds for seven periods, as shown in Figure 9.2(b), from 6.57 points and 0.299 at $m = 3$ to 2.92 points and 0.351 at $m = 30$. Each point is a mean over very different paths. Seed 47's funding SD is between 0.699 and 0.726 at every period, because its fund runs into surplus where the contribution sits on its 5 % floor, and seed 11's falls from 0.195 to 0.178 as m rises.

Case 3 replaces the recomputed spread with layers. On seed 11 the layers give the smoothest contribution of the three rules, 1.92 points against 2.58 for the spread and 2.04 for the PI, and the lowest funding, 0.724 at year 30 against 0.797, with a recovery plan started by the valuation of year 30. On the five seeds the order of the contribution SDs reverses, 5.86 points for the layers against 4.33 for the spread and 5.46 for the PI, and the lowest valuation averages 0.638 against 0.653, in line with the finding that spreading is more efficient than amortizing each gain and loss [12]. A PI with $\tau_I = 15$ yr behaves much like the layers, likely because its integral also keeps a deficit on the books after the proportional term has let it go.

9.6.3 The Late Valuation

Case 4 starts a deterministic fund at 0.9 with $m = 5$. With the valuation a year late the contribution climbs from 10 to 13.0, 16.1 and 17.6 points over the first three decisions and the ratio reaches 0.998

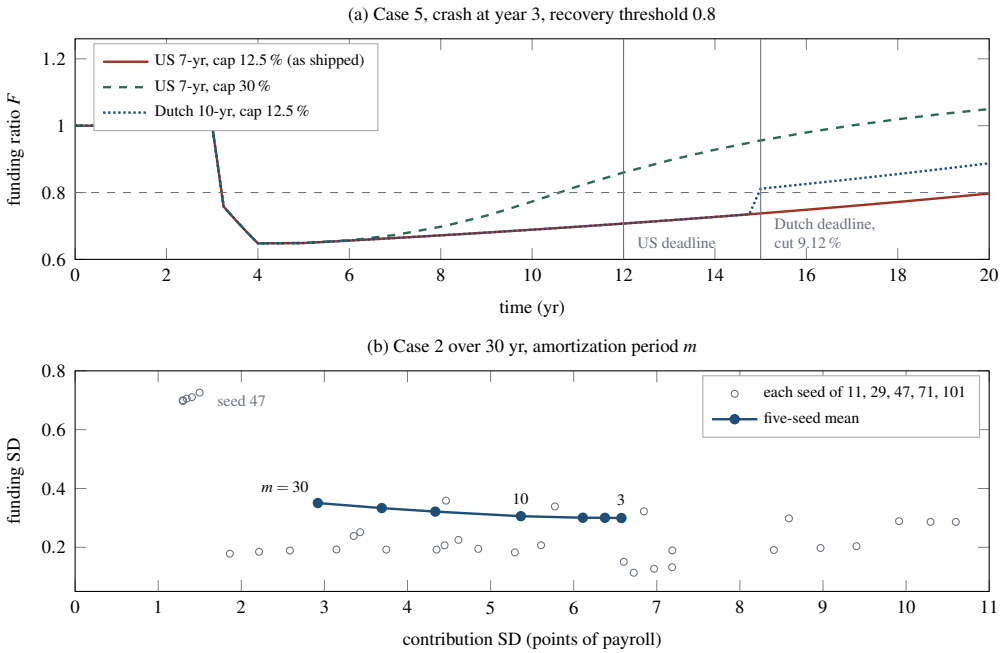


Figure 9.2 Authority and trade-off measured on the application's engine. (a) Case 5, deterministic, under the seven-year rule at the shipped 12.5 % cap (solid), at a 30 % cap (dashed), and under the ten-year Dutch preset (dotted), which cuts the liabilities by 9.12 % at year 15, where the run pauses until Run is pressed again. (b) Contribution SD against funding SD over 30 years for amortization periods of 3, 5, 7, 10, 15, 20 and 30 years, five-seed means (filled circles on a solid line) and each seed (open circles).

at year 15 without overshoot, against 0.994 with no delay. The delay raises the contribution SD from 1.91 to 2.35 points and to 2.93 at two years, where the ratio peaks at 1.003 at 12.02 yr. At $m = 3$ the delayed rule overshoots to 1.004 at 8.08 yr with an SD of 3.85, while without delay it does not, which matches the complex roots of Equation (9.7) below $m = 3.85$. At $m = 20$ the ratio ends at 0.933. The lag is a cost in contribution variability more than in the speed of recovery.

> ✦ **DID YOU KNOW? A Later Valuation, an On-Time Recovery**
>
> From the simulator: in case 4 the deterministic fund ends at 0.994 with no valuation delay, 0.998 with a delay of one year and 1.002 with two, as Table 9.5 prints. The delay does not slow the recovery. It shows in the contribution instead, whose SD rises from 1.91 to 2.35 and 2.93 points.

🕮 **Pause.** Try Exercise 9.2 before reading on.

9.6.4 A Recovery the Sponsor Cannot Fund

Case 5 runs the crash on a deterministic fund. The valuation of year 4 still reads 0.999, the one of year 5 reads 0.648 and starts the seven-year clock, and from the decision at year 5 the contribution sits on its 12.5 % cap. The deadline is judged by the valuation of year 12, which reads 0.698 while the fund stands at 0.707, as shown in Figure 9.2(a), and the ratio is 0.738 at year 15. Every amortization period from 3 to 30 years misses the same deadline, ending at 0.7379 for 3 and 5 years and 0.7376 from 7 to 30. At a 30 % cap the rule climbs by its move limit, three points a year, from 12.8 points at year 5 to 25.0 at year 9.04 and 25.4 at year 10, the valuation of year 12 reads 0.819 and clears the

Table 9.6 The Stochastic Cases on the Five Declared Seeds, With Means of the Final Ratio, the Lowest Published Ratio, the Contribution SD and the Mean Contribution, and the Seeds Whose Valuation Fell Below 0.8

Configuration	$F(T)$	F^{min}_{pub}	σ_c	$\bar{c}$	< 0.8
1, MANUAL at 10 %	0.783	0.686	0.00	10.00	3
2, $m = 7$	1.326	0.695	6.11	14.17	3
2, $m = 30$	1.051	0.598	2.92	10.83	3
3, layers	1.277	0.638	5.86	13.44	4
3, amortization	1.168	0.653	4.33	12.28	3
3, PI	1.269	0.653	5.46	13.41	4
6, $e = 0.9$	0.873	0.637	5.47	13.15	4
6, $e = 0.3$	0.826	0.707	6.19	16.35	5
7, glide on	0.882	0.705	4.76	10.78	3
7, glide off	1.503	0.798	4.12	10.22	3
8, holiday	0.871	0.652	6.80	9.46	3
9, MPC, $\kappa = 2$	0.917	0.775	4.12	17.46	3
9, MPC, $\kappa = 0$	0.781	0.673	4.40	13.67	4
9, US 7-yr	0.832	0.686	5.96	14.82	4

Seeds 11, 29, 47, 71 and 101, each run for its case's length.

clock, and the ratio is 0.956 at year 15. Under the Dutch preset the valuation of year 15 reads 0.727, and the cut that brings it to 0.8 reduces the liabilities by 9.12 %, which leaves the fund at 0.812. A deadline sets when the shortfall is counted, and the cap decides whether it can be met.

> **TRY IT IN THE APP** **5 · A Recovery the Sponsor Cannot Fund**
>
> Load the scenario and press Run. Watch the Valuation badge: after the crash of year 3 it still reads 0.999 at year 4 and drops to 0.648 only at year 5. From that decision the contribution sits on its 12.5 % cap, the valuation of year 12 reads 0.698, and the seven-year deadline is missed.

Pause. Try Exercise 9.1 before reading on.

9.6.5 Equity, the Glide Path and the Holiday

Case 6 holds 90 % in equities through the crash. On seed 11 it ends at 1.499 with a lowest valuation of 0.777, while 30 % in equities ends at 0.994 with 0.710 and needs a mean contribution of 14.2 points against 9.2. Once the discount rate has fallen to 0.025 yr, Equation (9.6) at the normal cost lets the ratio rise on its own only above 0.893 with an expected return of 0.028 yr, against 0.481 at 0.052 yr, so the low share needs contributions above the normal cost just to stand still. On seed 29 the order reverses, 0.469 at 90 % against 0.743 at 30 %, with lowest valuations of 0.395 and 0.645. Over the five seeds the high share widens the outcomes, final ratios from 0.351 to 1.499 against 0.658 to 0.994, with four seeds below 0.8 at some valuation against five for the low share. Neither allocation keeps every seed above the threshold, and one favorable seed shows nothing about the others.

Case 7 starts at 1.25 with the glide path on. On seed 11 the share falls to 0.35 at the start, the fund drifts down to 0.796 at year 30 with a mean contribution of 6.7 points, and with the glide path off it ends at 1.088 on 5.1 points, after a surplus that peaks at 2.005 at 14.42 yr while the contribution sits on its floor. The five-seed means are 0.882 against 1.503 at the end, with three seeds below 0.8 either way. The cost of de-risking is expected return, and on these seeds the reduced exposure does not buy a higher lowest valuation, 0.705 against 0.798 on average.

Case 8 imposes a five-year holiday on a fund at 1.25 and a crash at year 5. The contribution is zero until the decision at year 5, the floor then sets it to 5 points at once, and the rule raises it through 5.8, 8.9 and 11.5 to 13.2 points at year 10, the step from 5.8 to 8.9 at the move cap of 3.1 points. Run on the engine without the scripted holiday, the same seed holds 1.905 at year 5 against 1.819, after 0.070 initial liabilities of contributions, and ends at 1.210 against 1.165 with a lowest valuation of

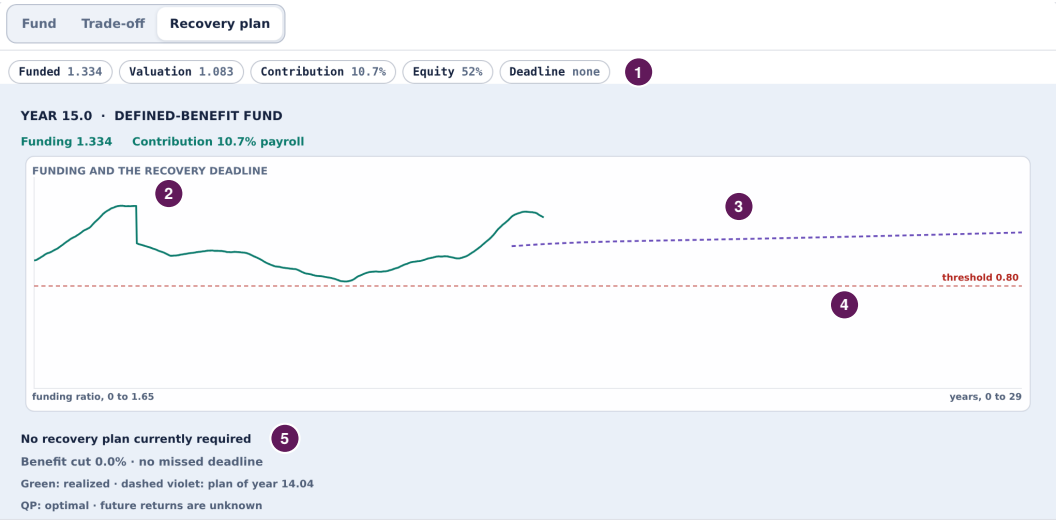


Figure 9.3 Case 9, the MPC with a funding buffer, at $t = 14.98$ yr (the clock is cut, the panel reads year 15.0) in the Recovery plan view, left in AUTO as shipped on return seed 11, cropped to the plant panel. (1) The badges read funded 1.334, valuation 1.083, contribution 10.7 %, equity 52 % and no deadline, so the current and the published ratio differ. (2) The realized ratio, with the crash at year 3, stays above (4) the labeled threshold of 0.80 throughout. (3) The dashed plan is that of the decision at 14.04 yr. It starts from the projected estimate of 1.110 and rises to 1.218 at year 29. (5) The status lines report no recovery plan, no benefit cut and no missed deadline.

0.922 against 0.879. On every one of the five seeds the holiday lowers the final ratio, by 0.016 to 0.093. A surplus spent as a holiday is not there when the crash arrives.

9.6.6 A Plan on an Estimate

Case 9 hands the decision to the MPC with $\kappa = 2$ and the glide path on, as shown in Figure 9.3. Its estimate knows the fall in the discount rate and the revaluation it causes, but not the equity loss. In case 5 with the MPC selected, the estimate reads 0.818 at 3.25 yr against a true 0.774 and a published 1.013. The MPC has held the contribution at the 12.5 % cap since its first decision, because its buffer on the declared $\sigma = 0.16$ sets a target of 1.192 in this deterministic case, while the seven-year rule first moves at year 5. The valuation of year 5 then brings the estimate to the true 0.675. In case 9 on seed 11 the estimate reads 1.153 at 4.5 yr against a true 1.052 and a published 1.423, and over the run it misses the true ratio by a root-mean-square 0.132, and the published valuation by 0.184. With the buffer the MPC ends at 1.331 with a lowest valuation of 0.839 and an SD of 2.85 points, without it at 1.126, 0.739 and 3.52, and the seven-year rule on the same draws at 1.245, 0.773 and 4.40. On the five seeds the buffer raises the lowest valuation from 0.673 to 0.775 on average and the mean contribution from 13.67 to 17.46 points, and no recovery deadline is missed with it, against two without.

> **TRY IT IN THE APP** **9 · A Certainty-Equivalent Recovery Plan**
>
> Run the scenario with the funding buffer of 2, then set Funding buffer to zero and press Reset. With the buffer the MPC ends at 1.331 with a lowest valuation of 0.839. Without it the same draws end at 1.126 with a lowest valuation of 0.739, and the contribution SD rises from 2.85 to 3.52 points.

Pause. Try Exercise 9.3 before reading on.

Table 9.7 The Challenge at Year 15 on the Five Declared Seeds

Seed	Policy	$F(15)$	F^{min}_{pub}	σ_c	total	Target
11	shipped	1.379	0.863	3.91	0.456	SD
29	shipped	0.537	0.478	6.44	0.904	missed
47	shipped	1.286	1.014	3.67	0.359	SD
71	shipped	0.779	0.787	4.55	0.471	floor, end, SD
101	shipped	0.549	0.640	6.75	0.838	missed
11	suggested	1.158	0.954	1.46	0.947	met
29	suggested	1.039	0.940	1.05	0.998	met
47	suggested	1.154	0.980	1.90	0.932	met
71	suggested	1.083	0.976	1.46	0.945	met
101	suggested	1.045	0.941	1.19	0.978	met

Total in L_0. Missed: a deadline, the floor, the end ratio, the SD.

9.6.7 The Challenge on Five Seeds and on a Hundred

The challenge starts a fund at 1.25 with the seven-year rule, a 25 % cap and 20 % of payroll in force, and adds a fall of one point in the discount rate over years 4 to 8 to the crash at year 3. Judged at year 15 on the annual calendar, it asks for every valuation at or above 0.8, a current ratio of at least 1.0, contributions at most 25 % of payroll with an SD of at most 3 points, and no cut. Credit is judged on the case's own seed 11, since a seed, a model parameter or a disturbance changed from the case voids the attempt. There the shipped rule fails on its SD of 3.91 points, 0.91 above the limit, with a lowest valuation of 0.863 and a final ratio of 1.379, and the suggested settings pass with a lowest valuation of 0.954, a final ratio of 1.158, an SD of 1.46 points and a largest contribution of 21.0 %. Table 9.7 gives every declared seed. The shipped rule fails on all five, on its SD alone on seeds 11 and 47, and with missed deadlines and the cap reached on seeds 29 and 101, because a fund at 1.25 lowers its contribution toward the 5 % floor through a crash its valuations do not yet show, reaching 5.0 points at year 4.04 on seed 11, and must then climb back by about three points a year. The suggested settings run the MPC with $F^* = 1.2$, $\kappa = 3$, $w_c = 4$, a glide path from 0.2 to 0.05 in equities over funding of 0.6 to 1.1, and 22 % in force at the start. They pass on all five seeds, ending between 1.039 and 1.158 with lowest valuations between 0.940 and 0.980, by contributing 18.7 to 19.9 points of payroll on average, nearly twice the normal cost, and 0.932 to $0.998\,L_0$ in total against 0.359 to 0.904 for the shipped rule.

Five seeds are a finite sample, so a hundred more were run on the same path. On seeds 1 to 100 the suggested settings pass 98 times, the two failures, seeds 83 and 98, exceeding the SD limit at 3.06 and 3.18 points with every other target met, while the shipped rule passes once, on seed 58. The final ratios of the suggested runs lie between 1.021 and 1.272 and their lowest valuations at or above 0.877.

The comparisons of Table 9.6 keep their direction on seeds 1 to 100. There the thirty-year spread averages a contribution SD of 2.39 points against 5.20 and a lowest valuation of 0.696 against 0.744, and misses a deadline on 39 seeds against 20. Equities at 90 % average a lowest valuation of 0.527 against 0.657 at 30 %, and the MPC's buffer raises it from 0.600 to 0.666 while the seeds with a missed deadline fall from 35 to 17. The layers keep a higher contribution SD than the spread, 4.61 against 3.59 points, at nearly the same average lowest valuation, 0.721 against 0.723. Figure 9.4 draws the five-seed means of Table 9.6 as a chart.

9.6.8 Deciding More Often

The calendar changes little here, because the valuation stays annual. In case 4 the contribution SD is 2.35 points on the annual calendar, 2.42 on the quarterly one with 60 decisions, and 2.46 with a decision at each of 156 executions, and the ratio ends at 0.998 on all three. With the credit χ of Equation (9.8) a finer calendar ramps as fast as the annual one. In case 5 at a 30 % cap the contribution rises over years 6 to 9 at 3.02, 2.99 and 3.02 points a year on the three calendars, and all three meet

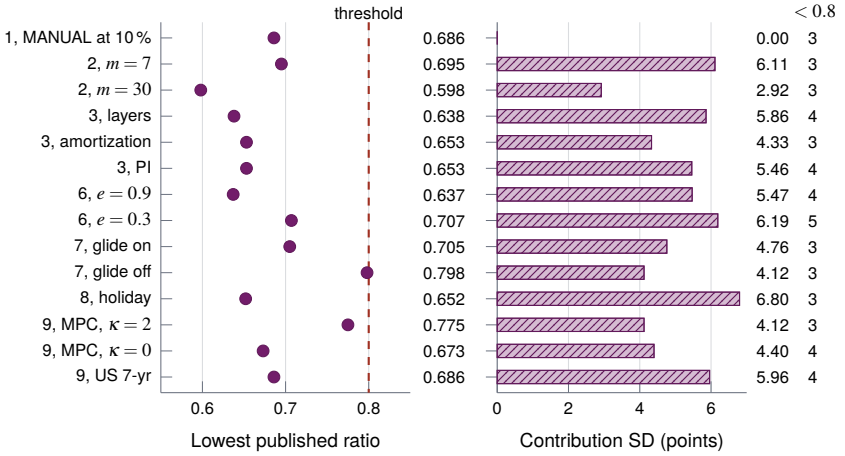


Figure 9.4 The stochastic cases of Table 9.6 as a chart: the five-seed mean of the lowest published ratio (dots, left, against the recovery threshold of 0.8, dashed) and of the contribution SD (hatched bars, right), with the number of seeds whose valuation fell below 0.8 at the far right. Compare rows within one case, because the cases start from different funds and disturbances. Notice that within case 2 the thirty-year spread halves the SD and lowers the lowest valuation, and that within case 9 the buffer raises the lowest valuation from 0.673 to 0.775 at a lower SD.

the deadline of year 12. For the MPC of case 9 a finer calendar lowers the five-seed contribution SD from 4.12 to 3.74 and 3.49 points but also the average lowest valuation, from 0.775 to 0.762 and 0.729, with one deadline missed on each finer calendar, likely because the same ten move blocks then leave the plan free only over its first 2.25 years or 0.87 years. The challenge accepts only the annual calendar.

9.6.9 The Matched Reference

The reference case holds a deterministic fund at the matched point. The ratio stays at 1.000 for fifteen years at the normal cost while both stocks grow at 0.015 yr. Pressed at year 0 before Run, the rates event lowers the ratio to 0.784 at year 5, from which the amortization rule brings it to 1.025 at year 15, and the longevity event leaves it at 0.976.

9.7 Technology in Practice

What Measures It

No instrument reads a funding ratio. The assets have a market value, but the liabilities are a present value that the actuary computes from the promised benefits and a discount rate, and the valuation brings both to one date. In United States law that date, the way assets may be averaged and the yields that discount the liabilities are all written into the minimum funding rules.

> **TECHNOLOGY** **Sensor: The Actuarial Valuation**
>
> The sensor of a funding loop is the actuarial valuation. Under the 2006 act every determination for a plan year is made as of its valuation date, the first day of the plan year for all but the smallest plans. Assets count at fair market value or at an average of market values reaching back at most to the last day of the 25th month before the valuation month, never below 90 or above 110 percent of market value. Liabilities are discounted at segment rates from a corporate bond yield curve that averages monthly yields over 24 months [9]. Both averages are filters: on a trend the actuarial ratio lags a crash and stays inside its corridor. The valuation of this chapter has no filter and reads market values exactly, a year late.

The valuation of Equation (9.4) stands for this instrument without its filters. It samples both stocks once a year and publishes them a delay θ_v later, and the projected estimate carries the published ratio forward through the recorded flows.

What Moves It

Two handles act on the fund. The contribution is cash from the sponsor, bounded by the sponsor's capacity, and the equity share sets how much of the market return, and of its risk, reaches the assets. In the chapter the contribution passes the quantum, the move limit and the bounds of Equation (9.8) before it reaches the fund, and a cap below the normal cost plus the deficit's spread saturates it, as in case 5.

> **TECHNOLOGY** **Final Element: The Sponsor's Contribution**
>
> The final element is the sponsor's cash contribution. Under the 2006 act the minimum required contribution for a plan year is due $8\frac{1}{2}$ months after the year closes, and a payment made on a date other than the valuation date is adjusted for interest. A plan with a funding shortfall for the preceding plan year pays four required installments, each 25 percent of a required annual payment, due on April 15, July 15, October 15 and January 15 of the following year. An installment paid short is charged interest at the plan's rate plus 5 percentage points [9]. The law thus fixes when cash moves. In the chapter the contribution flows in at the rate in force on every weekly step, and it changes only at the decisions of its calendar.

Where the Controller Runs

The platform of a funding rule is a calendar rather than a computer: the plan year, its valuation date, the actuary's certification and the dates on which contributions fall due. In the application the controller executes every $5/52 = 0.0962\,\mathrm{yr}$, but the shared decision calendar lets the contribution and the equity share move only at its instants, once a year by default. The layer that holds the chapter's structure is that calendar with its shaping, the rules above it and the projected estimate below them. The MPC solves its quadratic program at each decision, from the estimate rather than from the published ratio. Figure 9.5 draws the loop with these devices in place of the schematic's symbols.

What Is Modernizing It

Three developments bear on the late measurement. The first is stochastic projection. The United States insurer of private defined-benefit plans projects its programs with two models that share distributions of investment returns, interest rates and other variables, one running 500 scenarios and the other 5,000, and reports averages and ranges, calling the results estimates, not predictions [17]. The chapter's five declared seeds and its hundred more are the same idea at the scale of one fund. The second is a model corrected by late data: in that report the multiemployer plans' assets reported in the filings for 2022 were higher than the previous year's model had projected, and the new data raised that program's projected net position [17], much as the chapter's estimate restarts from each valuation. The third is a less filtered reading: the 2006 act lets a sponsor determine the minimum contribution from the full corporate bond yield curve without its 24-month averaging, an election revocable only with the consent of the Secretary of the Treasury [9].

> **IN PRACTICE** **Acting on a Ratio Not Yet Certified**
>
> The 2006 act says what a plan assumes while its new ratio is unmeasured. Its benefit limitations key on a funding percentage that the plan's enrolled actuary certifies. If a limitation applied in the preceding plan year, the ratio is presumed unchanged from that year until the actuary certifies the new one. If no certification is made before the first day of the 10th month of the plan year, the ratio is conclusively presumed to be below 60 percent. A plan that stood within 10 points of a limitation is presumed, from the first day of the 4th month, to be 10 points below its preceding-year ratio [9]. The statute holds the last reading, then assumes worse as that reading ages.

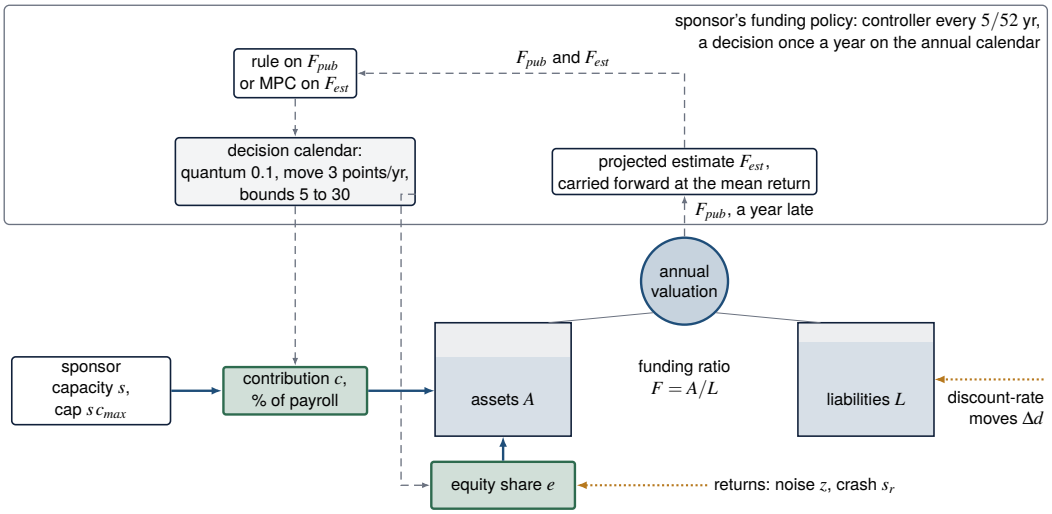


Figure 9.5 The funding loop of Sections 9.3 and 9.5 with a fund's real instruments. Green boxes are the final elements: the sponsor's contribution and the equity share. The blue circle is the measurement, the annual actuarial valuation of both stocks, published a year late. Solid arrows carry money, dashed arrows signals, and dotted arrows the market returns and discount-rate moves that disturb the fund. Notice that the controller runs many times a year, but the contribution and the equity share move only at the decisions of the calendar, and every decision rests on a reading a year old.

9.8 Limitations

The fund is an aggregate with payroll, accruals and benefits proportional to its liabilities, its assets carry no duration, and its returns are colored noise with a one-year correlation, so its five-year risk exceeds that of independent annual returns. The legal presets are simplified. The United States preset spreads the whole deficit over seven years where the 2006 act amortizes each year's shortfall in its own layer, the Dutch preset cuts at the first missed ten-year deadline rather than after five annual determinations below the minimum, and the threshold of 0.8 is a teaching value rather than either text's trigger for a recovery plan. The estimate observes no realized equity return, which keeps the valuation's delay in force for the MPC but discards what a fund knows of its own market value.

> **OPEN CHALLENGE** **A Sponsor Whose Capacity Falls With the Market**
>
> A funding rule has one handle, the contribution, and its size is uncertain. In multiemployer plans, contributions are driven mainly by hours worked and negotiated rates, so a fall in covered work lowers them and can make the minimum requirements hard to meet. Declining work after the 2008 to 2009 financial crisis was a significant factor in the crisis of those plans, and the insurer's projections for fiscal year 2024 still call future contribution levels uncertain and a significant risk [17]. The chapter models the sponsor's capacity only as a slider and a scripted hardship. Progress would be a funding rule judged on paths in which the cap falls with the market that drains the fund.

Every threshold, deadline, cap and challenge target is a teaching value, and five seeds, or a hundred, estimate no probability of success. The application and this chapter give no investment or pension advice.

9.9 Conclusion

This chapter studied a defined-benefit fund written as two stocks whose ratio drifts away at 0.025 yr when left alone and whose valve moves it by 0.0025 a year per point of payroll. Amortization is a

proportional gain with pole $0.025 - 1/m$, and a valuation a year late raises the contribution SD of the deterministic case at $m = 5$ from 1.91 to 2.35 points, while overshoot appears only below 3.85 years. After a crash to 0.648, a 12.5 % cap makes the seven-year deadline impossible for every period, and the Dutch preset closes the gap with a cut of 9.12 %.

On five seeds a thirty-year spread halves the contribution SD of a seven-year one, 2.92 against 6.11 points, and lowers the average lowest valuation from 0.695 to 0.598. The challenge's MPC with a buffer and a glide path passes on all five declared seeds and on 98 of a hundred by contributing nearly twice the normal cost.

The fund is an aggregate with simplified laws. A fuller model would add bonds whose price moves with the discount rate, cohorts with mortality, the statutes' layered bases and multi-year tests, and an estimate that weighs the valuation against the fund's observed market value.

The application turns the trade between the sponsor's volatility and the fund's risk into a measurement on seeded paths.

KEY POINTS

- A pension fund left alone drifts away from full funding, and a point of payroll moves its ratio by only 0.0025 a year.
- A valuation a year late costs contribution variability more than speed of recovery.
- A deadline sets when the shortfall is counted, and the cap decides whether it can be met.
- The projected estimate knows the flows and the discount rate but no realized equity return, so a crash reaches the plan only through the next valuation.

Where else this applies. Volume 2, Chapter 10 steers a public debt through the same annual calendar, and Volume 4, Chapter 8 a cash stock that drains faster than it refills.

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9.10 Exercises

9.1 Concept check · 2 min

In case 5 the crash takes a deterministic fund to a funding ratio of 0.648, and the sponsor's cap is 12.5 % of payroll. Every amortization period from 3 to 30 years misses the seven-year deadline and ends near 0.738. Why does the period make almost no difference?

(a) The year-late valuation starts every rule's recovery a year too late

(b) The move limit of three points a year slows a short period as much as a long one

(c) The quantum of 0.1 point rounds the faster rule's larger requests back to the slower one's

(d) Every period asks for more than the cap, so each applies 12.5 % from year 5 on

9.2 Concept check · 1 min

Case 4 starts a deterministic fund at 0.9 with $m = 5$. With the valuation a year late the ratio ends at 0.998 against 0.994 with no delay, and the contribution SD is 2.35 points against 1.91. What does the delay mainly cost here?

(a) Variability of the contribution, with almost no loss in the speed of recovery

(b) Speed of recovery, because every decision acts a year later

(c) An overshoot of the target, because the rule pays on an old deficit

(d) A steady offset, because a proportional rule on old data never reaches the target

9.3 Concept check · 2 min

In case 5 with the MPC selected, the projected estimate reads 0.818 at 3.25 yr while the true ratio is 0.774 and the published valuation 1.013. Why does the estimate sit between the two?

(a) The estimate averages the published valuation with the true ratio it receives each week

(b) The estimate knows the fall in the discount rate and the revaluation, but not the equity loss

(c) The funding buffer of the MPC adds $\kappa_e \sigma$ to the estimate it plans from

(d) The estimate integrates the realized return but holds the liabilities at the last valuation

9.4 Theory and calculation · 6 min

Use Equation (9.6) with $P = 0.25$, $n = 10$, $b = 0.05$, an equity share of 0.6, $r_f = 0.016$ yr and $\rho = 0.04$ yr, with no noise.

(a) Find g_L and the open-loop gain $r - g_L$ at the initial discount rate of 0.04 yr, and the time in which a deficit held at the normal cost doubles. *Checkpoint:* 0.015 yr, 0.025 yr and 27.7 yr.

(b) After the crash the discount rate is 0.025 yr. Find the new open-loop gain, the equilibrium ratio at a contribution of 12.5 %, and the rate at which the ratio climbs from 0.648 at that contribution. *Checkpoint:* 0.040 yr, 0.469 and 0.0072 yr.

(c) In case 5 the contribution stays at 10 % until the decision of year 5. Starting from 0.648 at year 4, find the ratio at year 5 and the year at which it reaches 0.8 at the 12.5 % cap. *Checkpoint:* 0.649 and year 20.2.

9.5 Theory and calculation · 6 min

The amortization rule decides once a year on a valuation a year old, and its deviation obeys Equation (9.7) with $\phi = e^{0.025}$ and $\gamma = (e^{0.025} - 1)/0.025$.

(a) Find ϕ and γ, and the roots at $m = 5$ with the delay and the single root without it. *Checkpoint:* 1.0253 and 1.0126, roots 0.758 and 0.267 against 0.823.

(b) Find the spread period below which the roots are complex, and the roots at $m = 3$. *Checkpoint:* 3.85 years, and $0.513 \pm 0.273\,j$.

(c) Relate the result to Equation (1.9). What plays the part of g, and why does the oscillation limit fall at g just above $1/4$ rather than at $1/4$? *Checkpoint:* $g = \gamma/m$. The growth factor $\phi > 1$ moves the limit to $\gamma/m = \phi^2/4$.

9.6 Theory and calculation · 6 min

Case 4 starts at $F_0 = 0.9$ with $m = 5$, a move limit Δ_c of three points a year, a quantum of 0.1 point and the contribution at 10 points.

(a) Find the amortization gain $100/(Pm)$ and the request c^w of Equation (9.9) at the first decision, and the contribution Equation (9.8) allows when a whole year's allowance is available. *Checkpoint:* 80 points per unit of ratio, 18.0 points, and 13.0.

(b) The second decision falls at 1.0577 yr and still sees a published 0.9. Find the allowance, the applied contribution and the credit χ carried forward. *Checkpoint:* 3.17 points, 16.1 points, credit 0.07.

(c) The third decision, at 2.0192 yr, applies 17.6 points without reaching its allowance. What published ratio does that imply? *Checkpoint:* About 0.905.

9.7 App challenge · 8 min

Select scenario 4, "A valuation that arrives late", at the default seed. Speed 1 runs it in about 15 s.

(a) Run 15 yr and read the final funding ratio and the Contribution SD tile of the scorecard. *Checkpoint:* 0.998 and 2.35 points (within 0.01).

(b) In Show Model set Valuation delay to 0, Apply, press Reset and run 15 yr again. Read the same two values. *Checkpoint:* 0.994 and 1.91 points.

(c) Repeat with Valuation delay 2. Read the SD, and explain from Equation (9.7) why the delay raises the SD more than it changes the final ratio. *Checkpoint:* 2.93 points.

9.8 App challenge · 8 min

Select scenario 5, "A recovery the sponsor cannot fund", at the default seed, in the Recovery plan view. Speed 1 runs it in about 15 s.

(a) Run 15 yr. Read the valuation of year 12 from the Published trend, and the final funding ratio. *Checkpoint:* 0.698 and 0.738 (within 0.003).

(b) Set Sponsor contribution cap to 30 %, press Reset and run 15 yr. Read the valuation of year 12, the final ratio and the largest contribution on the Contribution trend. *Checkpoint:* 0.819, 0.956 and 25.4 %.

(c) Set the cap back to 12.5 %, select Dutch-style recovery, press Reset and run until the run pauses at year 15. Read the Benefit cuts tile and the final ratio. *Checkpoint:* 9.1 % and 0.812.

9.9 App challenge · 8 min

Select scenario 9, "A certainty-equivalent recovery plan", at the default seed 11. Speed 1 runs it in about 15 s.

(a) Run 15 yr as shipped, with a Funding buffer of 2. Read the Lowest valuation and Contribution SD tiles and the final funding ratio. *Checkpoint:* 0.839, 2.85 points and 1.331 (within 0.01).

(b) Set Funding buffer to 0, press Reset and run 15 yr. Read the same three values. *Checkpoint:* 0.739, 3.52 points and 1.126.

(c) Set the buffer back to 2, select Historical US 7-year as the rule, press Reset and run 15 yr. Compare the three rules on these draws and say what the five declared seeds add. *Checkpoint:* 0.773, 4.40 points and 1.245.

9.10 Application area · 3 min

The fund holds no bonds, so a fall of 1.5 points in the discount rate revalues only the liabilities, through $D_L = 15\,\text{yr}$.

(a) Find the factor on L and the funding ratio of a fund at 1.0 after the revaluation alone, and after it together with the crash's equity factor of 0.807. *Checkpoint:* 1.252, 0.799 and 0.645.

(b) Outside the model, suppose 40 % of the assets were bonds whose value rose by the same factor and the rest did not move. Find the ratio after the rate fall alone, and name the parameter of Table 9.2 that this hedging would effectively change. *Checkpoint:* 0.879.

9.11 Application area · 4 min

A board asks the sponsor to restore a fund from 0.648 to full funding within seven years by contributions alone. The payroll is $0.25L$ a year and the normal cost 10 points.

(a) Ignoring the fund's growth and returns, how many points of payroll above the normal cost would that take each year, and what total contribution rate? *Checkpoint:* About 20.1 points, a rate of about 30.1 %.

(b) Compare with the case 5 cap of 12.5 %, which raises the ratio by at most 0.00625 a year, and with the application's largest cap. Name one cost to the sponsor that the model leaves out. *Checkpoint:* About 56 years at the cap, and above the 30 % slider maximum.

9.12 Review the agent · 5 min

An AI assistant reviews case 2 for a pension board. It writes, "On return seed 11 a thirty-year amortization period ends better funded than a seven-year one, 0.831 against 0.809, with a lowest valuation of 0.901 against 0.864 and half the contribution SD, 1.86 against 3.74 points. A long period is therefore both smoother and safer, and the board should adopt $m = 30$." Find the flaw, and say how you would check the claim in the application.