INTERACTIVE CONTROL ENGINEERING
Principles and Applications

Volume 2 Practical PID Control

CHAPTER 10

Phillips's Stabilization Policy, Budget Lags and a Folding Public Debt

Pages 247–271

Edited by John D. Hedengren

How to cite this chapter. "Phillips's Stabilization Policy, Budget Lags and a Folding Public Debt," in *Practical PID Control*, John D. Hedengren, Ed., vol. 2 of *Interactive Control Engineering: Principles and Applications*. Advanced Process Solutions, LLC, 2026, ch. 10, pp. 247–271.

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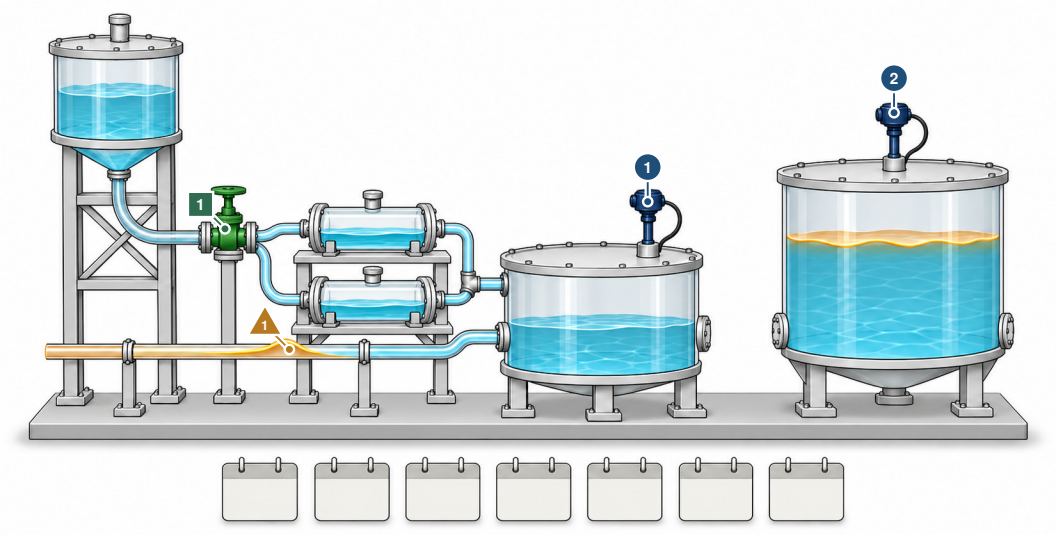
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CHAPTER 10

Phillips's Stabilization Policy, Budget Lags and a Folding Public Debt



The controller moves 1 the budget, decided once a year and delivered through two slow pipes, spending and transfers

The controller reads 1 the published output gap, read late and with errors 2 the public debt as a share of output

What disturbs it 1 a fall or rise in private demand

Notice the two delay vessels between the budget valve and the output tank: a budget set once a year reaches demand only through them, while the debt tank keeps every deficit. Illustration, not to scale.

IN PLAIN TERMS Steering an Economy With a Yearly Budget

The main idea. When people and firms spend less, a country makes less than it could. A government can then spend more, and some help comes by itself as taxes fall and benefits rise.

The challenge. The government sees a slump late and sets its budget once a year, so the money can come after the recovery and cause a boom. The extra spending is borrowed, and a debt whose interest outruns growth keeps growing by itself.

Why it matters. The timing of a budget decides whether it calms a slump or makes the swings larger.

THREE WAYS TO READ THIS CHAPTER

Explore	11 min	the first page, the history of the debt brake and the fact on interest rates (Section 10.2), the try-it segments of cases 4 and 6 and the fact on case 7 (Section 10.6), and the key points
Study	50 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	20 min	the first page, the results (Section 10.6), Technology in Practice (Section 10.7), Limitations (Section 10.8) and the open challenge

The control question A treasury decides its budget once a year on late figures, spends it the year after, and adds every deficit to a debt whose ratio to output has dynamics of its own. How much feedback do those lags and that calendar allow, and what does holding output up cost the debt?

Learning objectives

- Compute the multiplier and the offset that the automatic stabilizers leave as built-in proportional feedback.
- Explain why an annual budget halves the proportional gain that monthly decisions allow, and how a per-decision cap bounds the resulting cycle.
- Predict a debt path when the interest rate exceeds growth, and the offset of a proportional rule on the debt.
- Locate the fold of a debt with a risk premium, and judge whether a bounded consolidation can stop a spiral past it.
- Compare PID, rules and MPC on one crisis, and identify a constraint that acts procyclically.

Prerequisites Sections 1.4.2 (proportional offset), 1.6.2 (velocity form), 1.4.8 (derivative action), 1.4.9 (windup) and 1.4.11 (sample time). Section 1.6.3 is needed only for the predictive controller. Chapter 9 uses the same calendar, quantum and cap.

The app https://apmonitor.com/apps/fiscal-stabilization/

See also Volume 5, Chapter 9 for a stock refilled through a capped, year-late contribution. Volume 9, Chapter 4 for offset-free MPC with a disturbance state in the estimator, the role the private shock plays here.

10.1 Introduction

A recession that begins a quarter into the year barely shows in the treasury's figures when its budget is adopted that October. In the simulator of this chapter, private investment starts to fall at year 0.25, the true output gap is −1.15 % at the October budget of year 0.75, and the published gap reads only −0.51 %. A rule that asks for one point of stimulus per point of published gap raises the budget to 2.5 % a year later, the most one decision may add, and to 2.9 % at the October of year 2, when the published gap reads −2.89 % and the true one has already recovered to −1.46 %. The money reaches demand during the recovery. The slump's trough improves only from −6.23 % to −5.48 %, while the recovery, which peaks at +0.97 % with no budget at all, becomes a boom of +4.71 % at year 4.04.

Economists wrote this problem as feedback control before the vocabulary of control was common in their field. In 1954 stabilization policy was posed as proportional, integral and derivative correction of the gap between production and its desired level [1], in 1957 the same author asked what the lags of real policy do to that correction [2], and he had already built a hydraulic machine of the economy to show its dynamics [3, 4]. By 1973 it was a starting point that such policies could make a multiplier-accelerator model oscillate or go unstable, and stabilization was then set as a tracking problem in optimal control on a linearized model [5]. Two later results made the public debt part of the loop. The primary surplus of the United States rises with its debt ratio [6], which is a proportional feedback law on a measured state, and the stance of fiscal policy toward the debt was named active or passive in the same terms as monetary policy [7]. The feedback, its lags and the stock it accumulates are the subject matter, which is why control is the natural frame for it.

The structure is general. The controller acts through a decision calendar and a delivery lag, so its useful gain is bounded by delay as in any loop with dead time [8], and the integral of its own output is a stock with dynamics of its own, the debt, which here is an integrating state whose self-regulation changes sign. A plant with the same shape appears wherever holding one variable draws down a store, such as a battery's charge spent holding a vehicle's speed, a reservoir released to hold a flow downstream, or a propellant budget spent holding an orbit. Volume 4, Chapter 14 spends a vault of reserves to hold an exchange rate in its band. The belief this plant corrects is that a loop is finished once its error is gone. In the third case of the simulator an integral term brings the gap to zero against

a lasting fall in demand by holding the budget at 2.0 % of output for good, and the debt that pays for it climbs from 90 % to 111.6 % of output in twelve years. A second belief, that more feedback stabilizes better, fails too. On clean data, deciding once a year allows half the proportional gain that the same lags allow when the budget is decided every month, 0.76 against 1.53.

However, the work that estimates fiscal reaction functions, debt limits and interest-rate effects does so on historical data [6, 9, 10], where the budget calendar, the recognition lag and the rule are given. Such data rarely offer an economy in which a minister, Phillips's three terms, a debt rule, a brake and an optimizer each decide the same budget through one calendar and can be compared run against run. This chapter works with such an economy, built as a browser simulator, and measures what the lags, the calendar and the debt's own arithmetic do to each way of deciding. The multiplier-accelerator, Phillips's policy, Bohn's rule, the fold of a debt with a premium and model predictive control all come from the literature cited below. The simulator joins them in a continuous economy in which a budget passes through a recognition lag, a calendar with a quantum and a cap and two delivery pipelines before it moves output, while the debt integrates every deficit at an interest rate that depends on the debt. Every equation, constant and limit is stated in this chapter, and the app carries nine cases, a reference and a crisis challenge, each measured on the running engine.

The chapter is organized as follows. Section 10.2 traces the domain from the hydraulic machine to the debt brake. Section 10.3 writes out the economy, its budget pipelines and the debt ratio, and Section 10.4 describes its behavior before any decision, with the stabilizers as droop, the multiplier-accelerator cycle and the fold of the debt. Section 10.5 sets out the six ways the budget is set, from the minister's hand through Phillips's PID to a predictive controller, and Section 10.6 reports the measured cases, including the gain each calendar allows and the crisis challenge. Section 10.7 places the loop in the institutions of a real budget, Section 10.8 states the limitations, Section 10.9 summarizes the results, and the exercises follow in Section 10.10.

10.2 From Water Tanks to Debt Brakes

The oscillation the simulator starts from was written down in 1939, when the multiplier, which turns spending into income, was combined with the principle of acceleration, which turns the growth of income into investment [11]. Their interaction gives a second-order economy whose own response can oscillate, and its continuous form, with a production lag and an investment lag, is the plant of this simulator.

A decade later the dynamics were built in water. A. W. Phillips described mechanical models of economic dynamics in 1950 [3], and his machine was a hydraulic model of the British economy about 7 feet high, 5 feet wide and 3 feet deep, in which red water flowing through clear plastic tubes stood for the circular flow of income [12]. It used hydraulic components to simulate a dynamic system where other analog computers of the period used electrical or mechanical ones, and it showed the dynamic behavior of an economy to students and to professional economists alike [4]. The simulator's main view draws the same tanks and pipes, with the output tank read against a potential mark.

Phillips then wrote government into the machine as a controller. In 1954 the policy correction applied to the gap in production had three parts, one in proportion to the gap, one to its accumulated sum and one to the speed at which it was changing [1], the same three terms as the PID of Section 1.4.2, and in 1957 the paper on the time-forms of lagged responses asked how the shape of the lags between a decision and its effect changes what that correction does [2]. In 1973 Phillips's finding that such policies could produce undesired oscillations or instabilities in multiplier-accelerator models was the opening premise of a study that posed stabilization as a dual tracking problem in optimal control, with any limit on the states or the controls expressed as a higher cost on its deviation [5]. That treatment of limits as prices is the one the simulator's MPC uses for its deficit limit and its debt ceiling. The objection to such designs followed within three years. Relationships estimated under one policy can shift when the policy changes, because the people whose behavior they summarize respond to the policy [13], and that critique sets the first limitation of this chapter.

The debt entered the loop later. In 1991 a government's fiscal stance toward its debt was classified as active or passive alongside the stance of its central bank [7]. Evidence of corrective action followed in 1998. Standard tests cannot reject a unit root in the debt-to-output ratio of the United States, but its primary surplus is an increasing function of that ratio, so the ratio reverts to its mean once war-time spending and the cycle are controlled for [6]. Built-in stabilization was measured in the same years. Over 1962 to 1995 individual federal taxes in the United States offset perhaps as much as 8 % of initial shocks to output [14], which is the droop of Section 10.4 measured on a real tax code.

Rules came next. The Swiss debt brake of 2003, a ceiling on spending written into a constitution, is the subject of the history box below [15].

> ⧖ HISTORY **A Spending Ceiling in a Constitution, 2001**
>
> Swiss voters approved a debt brake for the federal government by a majority of 85 percent in a mandatory referendum in 2001, and it first governed the federal budget of 2003. The constitution requires income and expenditure to balance over time, caps total expenditure at revenue adjusted for the cycle, and requires spending above the cap in the final accounts to be made up in later years. Federal debt fell from 28.2 percent of output at the end of 2002 to 19.5 percent in 2012. A review after ten years found that the binding ceiling on expenditure was the rule's most important feature, and that the fall in nominal debt came in a favorable economy that the budgets had not anticipated, mainly through revenue and growth that were underestimated [15].

The price of borrowing joined the problem as a function of the debt. Estimates from projected deficits put the effect of one point of projected deficit-to-output ratio at roughly 25 basis points on long-term rates [9]. When a government's ability to raise its primary balance cannot keep pace with rising debt, the debt has an endogenous limit beyond which it cannot be rolled over, and data on 23 advanced economies from 1970 to 2007 show a fiscal reaction function of that shape [10]. The fold of this simulator is a static, deterministic cousin of that limit. The low interest rates of the 2010s reopened the other side. Safe rates below growth rates were shown to be more the historical norm than the exception for the United States, so that a debt may be rolled over without later taxes, while equilibria in which investors who believe a debt risky raise its cost by demanding a premium remain a relevant argument against high debt [16]. The fifth case of the simulator raises the interest rate above growth, and the sixth starts past the fold.

> ✦ DID YOU KNOW? **Rates Below Growth in Four Economies, 2019**
>
> When a lecture on public debt was published in 2019, 10-year government bond rates were about 3 percent in the United States, 1.3 in the United Kingdom, 1.2 in the euro area and 0.1 in Japan, against forecasts of 10-year nominal growth of about 4, 3.6, 3.2 and 1.4 percent [17]. Case 5 reverses the sign.

10.3 The Machine, Its Budget Pipelines and the Debt

The simulated economy is closed and real, and every flow and stock is a fraction of potential output, a year's worth for a flow. Output Y is a fraction of potential, so the output gap is $y = Y - 1$, and time is in years. The economy is drawn in Figure 10.1 in the manner of the application's MONIAC view, and Table 10.1 classifies its signals.

10.3.1 Production, Households and Demand

Production follows demand through Phillips's production lag, and households spend a share of disposable income, as shown in Equation (10.1).

$$\begin{aligned} \tau_p \dot{Y} &= D - Y, \qquad D = C + I_{tot} + G + X + d \\ C &= c\,(Y - T + tr), \qquad T = \bar{T} + t\,y \\ tr &= tr_0 - s_{tr}\,y + G_t + tr_{ex}, \qquad G = G_0 + G_s \end{aligned} \tag{10.1}$$

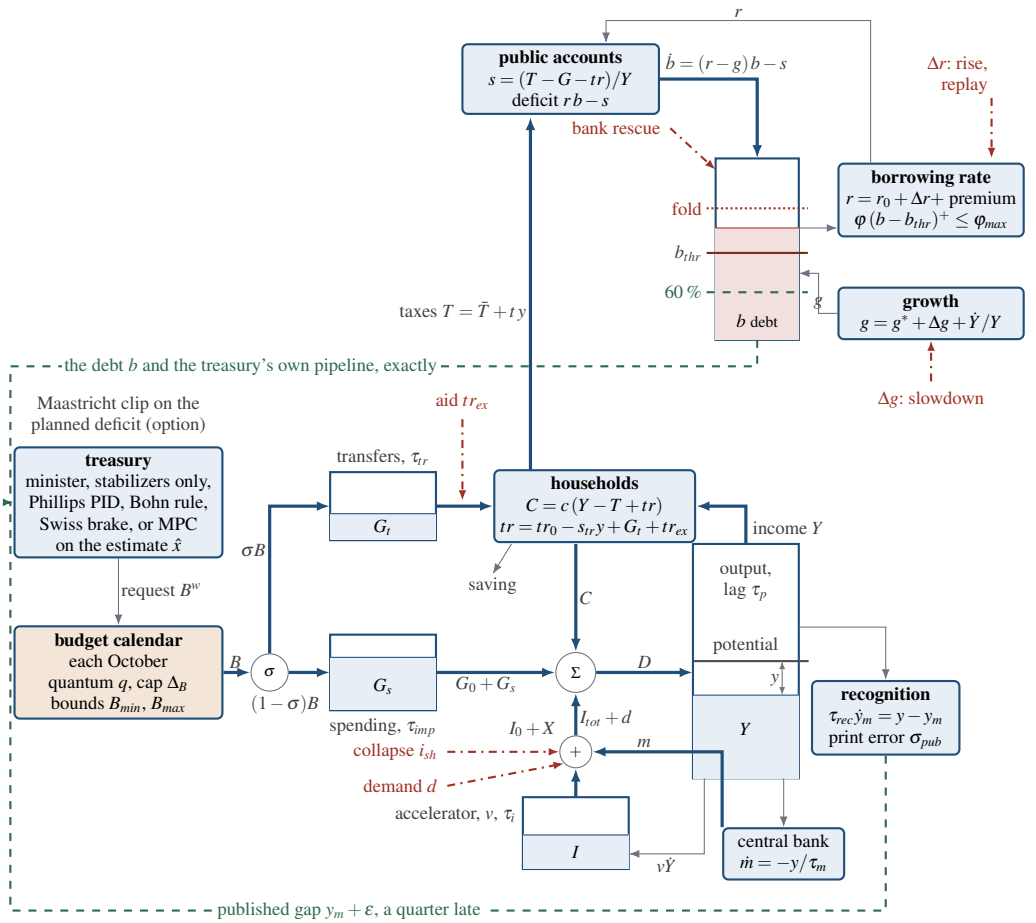


Figure 10.1 The economy as Phillips drew it, not to scale. Demand D, the sum of consumption C, government spending $G_0 + G_s$, private demand $I_{tot} + d$ and the constant X, fills the output tank, which is read against potential, so the gap y is the distance between the level and the mark. Income Y returns to households, which pay taxes T to the public accounts, receive benefits tr and spend C. The accelerator's induced investment I and the central bank's offset m join private demand. The treasury's request B^w passes the calendar's quantum q, cap Δ_B and bounds, and the budget B splits into spending G_s and transfers G_t, each with its own delivery lag. The primary balance s and the borrowing rate r fill the debt ratio b, marked with the 60 % reference (dashed), the premium threshold b_{thr} (solid) and the fold (dotted). Red dash-dotted arrows are disturbances, and the dashed path is all the treasury sees.

where $\tau_p = 0.25\,\text{yr}$, $c = 0.75$, $\bar{T} = 25\,\%$ of output is the tax revenue at potential, $t = 0.25$ the marginal tax rate, $tr_0 = 5\,\%$ the benefits paid at potential, $s_{tr} = 0.3$ the benefits added per point of negative gap, and $G_0 = 20.9\,\%$ the government spending at potential. The constant $X = 1 - c\,(1 - \bar{T} + tr_0) - I_0 - G_0 = 2.1\,\%$ is fixed so that demand equals potential output at $Y = 1$ with no policy and no shock. The disturbance d is other private demand, a lasting shift from a slider or the lockdown of a pandemic, and tr_{ex} is exogenous transfers, the emergency aid of a pandemic or a demographic ramp. The terms t and s_{tr} are the automatic stabilizers. They act at once and without any decision, so they are feedback built into the plant.

Table 10.1 Signals of the Economy and the Treasury by Role

Symbol	Role	Default	Range
y	CV, the true output gap, judged but never seen	0	—
$y_m + \varepsilon$	measured, the published gap, PV of the PID	0	SP 0
b	CV of the debt rules, measured exactly	90 %	start 0–200 %
B	MV, the discretionary budget in force	0	−5–8 %
B^w	request, from the minister's slider or a controller	0	−6–10 %
i_{sh}, d	DVs, investment shock, other private demand	0	events, ±6 %
tr_{ex}	DV, exogenous transfers (pandemic aid, ageing)	0	events
R	DV, a bank rescue added to the debt	0	10 %
Δr, Δg	DVs, borrowing-rate and growth shifts	0, 0	−2–6, −2–1
Y, I, m	states, output, induced investment, offset	1, 0, 0	—
G_s, G_t, y_m	states, spending and transfer pipelines, recognized gap	0	—
I_d, I_y	states, integrals of the deficit and of the gap	0	—
s, r, g, u	outputs, primary balance, rate, growth, unemployment	−0.9, 1, 2, 4 %	—

Flows and stocks in percent of potential output, rates in percent a year.

10.3.2 The Accelerator and the Central Bank

Firms invest in proportion to the growth of output, through a lag of their own, and an optional central bank adds private demand while output is below potential, as shown in Equation (10.2).

$$\tau_i \dot{I} = v\dot{Y} - I, \qquad I_{tot} = I_0 + I + m + i_{sh}, \qquad \dot{m} = -y/\tau_m \tag{10.2}$$

where $v = 0.6$ yr of output growth is invested, $\tau_i = 1$ yr, $I_0 = 17\,\%$ is autonomous investment, i_{sh} is the investment shock and $\tau_m = 4$ yr. With the offset switched off, $\dot{m} = 0$, which is Phillips's 1954 economy or a member of a currency union. The offset is the application's one addition to its original design. Without it a permanent cut of 1 % of output in spending leaves a permanent gap of –1.51 % and improves the primary balance by only 0.16 %, which the application's check measures, so no debt could be reduced without a lasting slump. The offset is an integral on the gap, not a model of monetary policy, and the borrowing rate does not follow it.

10.3.3 The Budget, Its Pipelines and Recognition

The decided budget B is held between decisions and reaches the economy through two first-order pipelines, while the treasury sees the gap through a recognition lag, as shown in Equation (10.3).

$$\begin{aligned} \tau_{imp}\dot{G}_s &= (1-\sigma)B - G_s, \qquad \tau_{tr}\dot{G}_t = \sigma B - G_t \\ \tau_{rec}\dot{y}_m &= y - y_m \end{aligned} \tag{10.3}$$

where $\tau_{imp} = 1$ yr is the implementation lag of spending, $\tau_{tr} = 0.25$ yr that of transfers, σ the share of the budget paid as transfers, zero by default, and $\tau_{rec} = 0.25$ yr the recognition lag. The published gap is $y_m + \varepsilon$, where ε is a print error with standard deviation σ_{pub}, zero by default and 0.5 % in the fourth case, drawn once a quarter from the engine's seeded generator so that a reset replays it.

10.3.4 The Debt Ratio

The debt ratio integrates the primary deficit at an interest rate less the growth rate, as shown in Equation (10.4).

$$\begin{aligned} \dot{b} &= (r-g)\,b - s, \qquad s = (T - G - tr)/Y \\ r &= r_0 + \Delta r + \min\left(\varphi_{max},\ \varphi\,(b - b_{thr})^+\right) \\ g &= g^* + \Delta g + \dot{Y}/Y \end{aligned} \tag{10.4}$$

where s is the primary balance, $r_0 = 1\,\%$ the real borrowing rate without a premium, $g^* = 2\,\%$ a year potential growth, $\varphi = 0.03$ points of premium per point of debt above $b_{thr} = 100\,\%$, capped at $\varphi_{max} = 10$ points, and Δr and Δg the shifts that a slider, an interest-rate rise of two points, a

Table 10.2 Constants of the Economy, With M for a Parameter of the Model Dialog and S for a Value Fixed in the Source

Symbol	Meaning	Value	Range	Class
c	propensity to consume	0.75	0.5–0.9	M
$\bar{T}$, tr_0	taxes and benefits at potential	25, 5 %	15–45, 0–15	M
G_0	spending at potential	20.9 %	10–35	M
I_0	autonomous investment	17 %	5–30	M
τ_p	production lag	0.25 yr	0.05–1	M
τ_i, v	investment lag, accelerator	1, 0.6 yr	0.25–3, 0–1.5	M
τ_m	central bank's integral time	4 yr	1–15	M
τ_{rec}	recognition lag	0.25 yr	0.02–1	M
τ_{imp}, τ_{tr}	spending and transfer lags	1, 0.25 yr	0.02–2, 0.02–1	M
σ_{pub}	print error, a new draw each quarter	0	0–1 %	M
b_0	debt at the start	90 %	0–200	M
r_0, g^*	rate without premium, potential growth	1, 2 %	−2–8, 0–5	M
φ, b_{thr}	premium slope, threshold	0.03, 100 %	0–0.3, 30–200	M
φ_{max}	largest premium	10 points	2–30	M
u^*, β	unemployment at potential, Okun	4 %, 2	2–8, 1–3.5	M
X	other autonomous demand	2.1 %	computed	S
—	investment collapse, boom	−2.5, +3 %	1, 1, 3 or 2 yr	S
—	pandemic lockdown, aid	−10.5, 10 %	one year	S
—	bank rescue, demographic ramp	10 %, 0.2 %/yr	20 yr	S
—	rate rise, growth slowdown	+2, −1 points	for good	S
—	bad quarter, deficit limit	±2 %, 3 %	—	S
—	market access lost	250 %	—	S
Δt	integration step, control period	1/48, 1/12 yr	—	S
—	log rate, seed	48 per yr, 20261005	—	S

growth slowdown of one point or the replayed policy rate of the inflation-targeting application make. The overall deficit, the quantity that the Maastricht limit of 3 % applies to, is $rb - s$. The ratios use $\max(Y, 0.2)$ in place of Y, a floor far outside any run. Unemployment is $u = u^* - y/\beta$, with y in percent, $u^* = 4\,\%$ and $\beta = 2$ points of gap per point of unemployment. Two integrals, $\dot{I}_d = rb - s$ and $\dot{I}_y = y$, give the exact calendar-year deficits and quarterly gaps, interpolated to the boundary inside a step. A quarter whose average true gap is outside ±2 % is a bad quarter, a calendar year whose deficit exceeds 3 % is a deficit year, two quarters of falling output raise RECESSION, and a debt above 250 % raises MARKET ACCESS LOST.

10.3.5 Shocks, the Start and the Integrator

Each event shock is a pure function of the time since it started. An investment collapse lowers i_{sh} linearly by 2.5 % over a year, holds it a year and recovers over three, and an investment boom raises it 3 % in the same shape with a recovery of two years. A pandemic sets $d = -10.5\,\%$ and $tr_{ex} = 10\,\%$ for one year. The demographic ramp adds 0.2 % a year to tr_{ex} for twenty years. A bank rescue adds $R = 10\,\%$ to the debt at the end of the step in which it fires, and a financial crisis is a collapse and a rescue together. A case starts at $Y = 1$ with every other state at zero and $b = b_0$, which is an equilibrium for the reference budget because $s = \bar{T} - G_0 - tr_0 = -0.9\,\%$ is exactly $(r_0 - g^*)\,b_0$ at $b_0 = 90\,\%$, with an overall deficit of 1.8 %, the growth rate times the debt. The application's check finds every derivative within 7×10^{-18} of zero and nothing moving over ten years. The engine of Section 1.6.1 advances the nine states by classical fourth-order Runge-Kutta in steps of $\Delta t = 1/48$ year, four to a month so that the monthly controller lands on the grid. It logs a row after every step, and the default seed is 20261005. Halving the step leaves both verdicts of the challenge unchanged and moves its final debt by 1.0×10^{-5} of output. Table 10.2 gathers the constants.

10.3.6 Assumptions

Expectations look backward. Households spend out of today's disposable income and do not anticipate the taxes a debt implies, firms invest on today's growth, and the premium is a fixed function of today's

debt. The model has no inflation and so no anchoring of expectations. This breaks down whenever the public learns the rule, which is the subject of Section 10.8.

Output is set by demand. Potential grows at g^* whatever happens, and only the central bank's offset brings output back to potential, as an integral that repays the area of a slump with a boom of the same area. An economy with a supply side or with prices that adjust would recover by itself.

One closed, real economy. There is no exchange rate, no inflation to erode the debt and no maturity structure, so the whole debt pays today's rate. The deficit is real, which makes its 3 % limit looser than a nominal one. The replayed policy-rate path is added one for one to the real borrowing rate as a stress test, although the rate in the inflation application rose because inflation did.

Teaching values. Every coefficient, lag, threshold and shock size is a declared teaching value. The premium slope of 0.03 points per point and the Bohn coefficient of 0.05 are orders of magnitude the application attributes to the empirical literature [6, 9], and none of them is estimated here.

10.4 What the Economy Does Before Anyone Decides

10.4.1 Stabilizers as Droop

At a steady state $\dot{Y} = 0$, so induced investment is zero and the gain from a lasting change in spending to output is the multiplier of Equation (10.5).

$$\frac{dY}{dG_s} = \frac{1}{1 - c\,(1-t) + c\,s_{tr}} = \frac{1}{0.6625} = 1.509 \tag{10.5}$$

Without the stabilizers it would be $1/(1-c) = 4$. The application's check measures 1.509 on the engine after thirty years of a step with the accelerator removed, and a budget paid entirely as transfers measures 1.132, which is c times the multiplier, because a transfer reaches demand only after households have spent their share of it. Each point of gap changes taxes by t and benefits by s_{tr}, so the stabilizers are a proportional controller from the gap to the net tax, of gain 0.55, built into the plant as a governor's droop is built into an engine. Like droop they leave an offset, the proportional offset of Equation (1.3). A lasting fall of 2 % in private demand settles at a gap of $-2 \times 1.51 = -3.02\,\%$ when nothing else acts.

🕮 **Pause.** Try Exercise 10.1 before reading on.

10.4.2 The Multiplier-Accelerator Cycle

Linearized at potential with no policy and no offset, output and induced investment have the eigenvalues $-0.625 \pm 1.503i$ per year, a damped cycle of 4.18 yr, and the application's check finds the same period within 0.003 years in the free response to a one-quarter demand impulse. The benefit response does much of the damping. Without it the pair moves to $-0.175 \pm 1.311i$. The original design of the application set the investment lag at half a year, where the same accelerator is unstable at zero policy, with eigenvalues $+0.075 \pm 2.301i$, so the lag was set to a year and the economy rings but settles.

10.4.3 The Debt's Own Dynamics and the Fold

With output at potential and no premium, Equation (10.4) is linear in b with the eigenvalue $r - g$. The check confirms it at -0.005, 0 and $+0.005$ per year for r_0 of 1.5, 2 and 2.5 %. While $r < g$ a debt fades under any fixed budget, and when $r > g$ it grows by itself. A fixed primary balance s holds the debt still where $h(b) = s$, with

$$\begin{aligned} h(b) &= \big(r(b) - g\big)\,b \\ h'(b) &= r_0 - g^* + \varphi\,(2b - b_{thr}), \quad b > b_{thr} \end{aligned} \tag{10.6}$$

the surplus that keeps each ratio constant at potential, and the debt is stable where $h'(b) < 0$. Below the threshold h is a line of slope -0.01. Above it the premium turns h up, and the same budget gains a second, unstable equilibrium, the fold, as shown in Figure 10.2(a). For the reference balance of

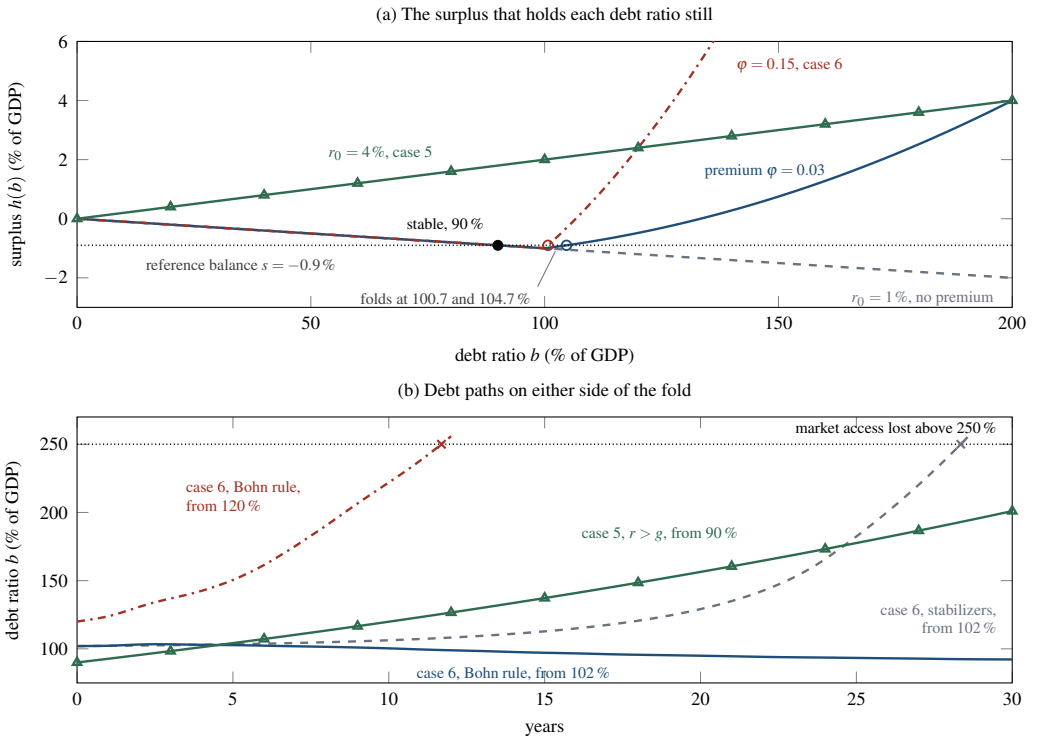


Figure 10.2 The debt's own dynamics, from the application's functions and engine. (a) The surplus $h(b)$ that holds each debt ratio still at potential, for the reference economy without a premium (gray dashed), with the shipped premium (blue solid), with the case-6 premium (red dash-dotted) and at $r_0 = 4\,\%$ (green with triangles), against the reference balance of –0.9 % (thin dotted). Each crossing is an equilibrium, stable where h falls through the line and a fold where it rises. (b) The debt ratio over thirty years for the case-6 economy from 102 % with the stabilizers alone (gray dashed) and with the shipped Bohn rule (blue solid), from 120 % with the rule (red dash-dotted), and for case 5 from 90 % (green with triangles), each start set in the Model dialog before Run. The crosses mark the loss of market access at years 11.69 and 28.35.

–0.9 % the equilibria are 90 % and 104.67 %, and the debt's eigenvalue at the fold is +0.023 per year. With the case-6 slope of 0.15 the fold moves down to 100.71 %, close enough to the threshold that a debt starting at 102 % with the stabilizers alone drifts slowly, to 106.3 % at year 10 and 129.1 % at year 20, and then runs away, losing market access at year 28.35, while one starting at 100 % falls slowly, to 97.4 % at year 30, as shown in Figure 10.2(b). Bisecting the starting debt on the engine with the stabilizers alone places the threshold at 100.71 %, the analytic fold. The original design's values, $r_0 = 2\,\%$ and a threshold at 60 %, would have put the reference debt of 90 % on the unstable branch, with $h'(0.9) = +0.036$, which is why the application moved both.

10.5 Six Ways to Set the Budget

The treasury's code runs once a month. At each run it draws a fresh print error if a quarter has closed, advances its estimate of the economy, obtains a request B^w from whichever controller is selected, clips that request when the Maastricht option is on, and then lets the calendar decide whether the month is a budget month. The minister and the automatic controllers share every step after the request. Table 10.3 lists the settings.

Table 10.3 Treasury Settings, Defaults and Ranges. A Marks an Advanced Control

Symbol	Setting	Default	Range
Budget and built-in feedback			
—	controller in AUTO	stabilizers only	5 controllers
—	calendar	each October	3 calendars
σ	transfer share of the budget	0	0–1
t, s_{tr}	marginal tax rate, benefit response	0.25, 0.3	0.1–0.5, 0–0.7
—	central bank closes the gap	on	on, off
q, Δ_B	quantum, largest change (A)	0.1, 2 %	0–1, 0.5–10
B_{min}, B_{max}	deepest cut, largest stimulus (A)	−5, 8 %	−10–0, 0–12
—	Maastricht deficit clip (A)	off	off, on
Phillips PID			
K_c	budget per point of gap	0.5	0–3
τ_I, τ_D	integral and derivative times	8, 0 yr	0–20, 0–2
Rules on the debt			
ρ_B, b^*	Bohn coefficient, debt target	0.05, 90 %	0–0.5, 0–150
K_{br}, T_{rep}	brake stimulus per point, repayment time	0.3, 3 yr	0–2, 1–12
η	brake band, share of spending (A)	0.06	0.02–0.2
MPC			
ν	move penalty	2	0.1–50
b_{ceil}	terminal debt ceiling	100 %	40–160
H	horizon (A)	8 yr	2–12
—	deficit limit (A)	strict	strict, soft, off
—	information (A)	estimator	estimator, truth

A disturbance slider shifts private demand by up to ±6 % of output.

10.5.1 Calendar, Quantum and Cap

Both the calendar and the rounding of each budget are those of `core/cadence.js`, a module of the framework of Section 1.6. The same module paces the price of Chapter 9 and the policy rate of Chapter 11. The annual calendar has one instant a year at $t = k + 0.75$, each October, the quarterly calendar one every quarter from $t = 0$, and the continuous calendar makes every monthly run a budget. A run becomes a budget if some calendar instant has passed since the previous budget, so an instant between two runs is taken at the later one. Every instant of the three calendars falls on a monthly run, so each budget is taken on time. The request then becomes a budget through Equation (10.7).

$$\begin{aligned} \tilde{B} &= q\left[B^{w}/q\right], \qquad \Delta_q = q\max\left(1, \lfloor \Delta_B/q \rfloor\right) \\ B_k &= \mathrm{clip}_{[B_{min}, B_{max}]}\left(B_{k-1} + \mathrm{sat}_{\Delta_q}\left(\tilde{B} - B_{k-1}\right)\right) \end{aligned} \tag{10.7}$$

where $[\cdot]$ rounds a tie away from zero, B_{min} and B_{max} are moved inward onto the grid before the clip, sat limits a move to $\pm\Delta_q$, and an undefined request leaves the budget where it was. With $q = 0.1\,\%$ and $\Delta_B = 2\,\%$, a budget moves in tenths of a point of output and by at most two points a year. The check finds every budget of nine runs, three calendars by three controllers through the crisis of case 8, on its calendar instant, on the grid, inside the cap and inside the bounds. A switch of mode or of controller during a run holds the budget in force at the next decision, and one made before the run starts does not.

10.5.2 The Minister and the Stabilizers

In MANUAL the request is the minister's slider, from −6 % to 10 % in steps of 0.1 %, and the calendar shapes it as it shapes a controller's. While a controller decides, the minister's slider shows what that controller requested before any rounding, capping or clipping, which the check confirms in case 6 at a Bohn coefficient of 1, with a request of −45.7 % against an applied −4 %. With the stabilizers selected the request is zero and only t and s_{tr} act.

10.5.3 Phillips's PID

The PID is the framework's shared controller in its velocity form, the algorithm of Section 1.6.2, run every month on the published gap against a set point of zero, as shown in Equation (10.8).

$$\begin{aligned} e_k &= -p_k, \quad P_k = K_c e_k, \quad D_k = -K_c \tau_D (p_k - p_{k-1})/T_c \\ \Sigma_k &= \Sigma_{k-1} + (P_k - P_{k-1}) + (D_k - D_{k-1}) \\ B_k^w &= \Sigma_k + \mathscr{I}_{k-1}, \qquad u_k = \mathrm{clip}_{[B_{min}, B_{max}]}\left(B_k^w\right) \\ \mathscr{I}_k &= \mathscr{I}_{k-1} + \left[K_c e_k/\tau_I + (u_k - B_k^w)/\tau_I\right] T_c \end{aligned} \tag{10.8}$$

where $p_k = y_{m,k} + \varepsilon_k$ is the published gap at execution k, $T_c = 1/12$ year, and $K_c = 0.5$ and $\tau_I = 8\,\mathrm{yr}$ by default. In the symbols of Equations (1.23) to (1.26) the action is $a = +1$, the bias is zero, Σ_k is S_k, B_k^w is v_k, $\mathscr{I}_k$ is I_k and the tracking time is τ_I, so only the output that leaves the controller is clipped and only the integral is corrected back toward it. A negative gap gives a positive error and a stimulus, which the application's direction check confirms with a request of +0.72 % at the monthly run one month after the recognized gap is set to −2 %. The request handed to the calendar is the unclipped B_k^w. The derivative is taken on the published gap without a filter, $\tau_I = 0$ removes the integral, and there is no back-calculation without one. The integral is not told about the quantum, the cap, the annual hold or the Maastricht clip, so it keeps integrating the gap between budgets, a form of windup that the back-calculation of Section 1.4.9 does not correct, since it acts only against the clip. While another controller or the minister decides, the PID resets $\mathscr{I}$ each month so that its output equals the budget in force B, which makes the switch to it bumpless, as in Section 1.4.10.

10.5.4 Bohn's Rule, the Brake and the Maastricht Clip

Bohn's rule is a proportional controller on the debt, $B^w = -\rho_B (b - b^*)$, so the primary surplus rises with the debt as the 1998 estimate found [6]. The Swiss-style brake allows a stimulus proportional to the published gap, debits a compensation account a with every budget above zero and credits it with every budget below, and never lets the account fall below its band, as shown in Equation (10.9).

$$\begin{aligned} B^w &= -K_{br}(y_m + \varepsilon) + \min(0, a)/\max(0.5, T_{rep}) \\ \bar{B} &= (a + A_b)/\max(1, T_d) \\ a &\leftarrow a - B_k T_d \quad \text{at each budget} \end{aligned} \tag{10.9}$$

where $A_b = \eta\,(G_0 + tr_0) = 1.55\,\%$ of output is the band, T_d the interval between budgets, one year on the annual calendar, and $\bar{B}$ the largest budget that keeps the account inside the band until the next one. A request above $\bar{B}$ is clipped to it, and a shaped budget above it is cut to the grid point below it, overriding the cap and the quantum. Credits are kept as debt repaid rather than spent again, so the bound is one-sided, and the brake is a bounded integrator, in which discretion is allowed and its sum is not. The check finds the account never below $-A_b$ in any case, at gains up to 2 and on the monthly calendar, with 0.054 % of output to spare at its closest. The Maastricht clip, when it is on, limits any controller's request to $B^w \le 0.03 - (r\,b - s - G_s - G_t)$, the budget that would keep the deficit within 3 % at today's revenue. The 60 % reference for the debt is drawn but not enforced.

10.5.5 The Treasury's Estimate

The MPC plans from an estimate $\hat{x}$, because the treasury does not know the private shock, the induced investment or the central bank's offset. Each month the estimate is stepped over T_c by fourth-order Runge-Kutta on the plant's own Equations (10.1) to (10.3), with the private shock $\hat{w}$ held, its own pipelines G_s, G_t and the debt b copied from the truth, and then corrected by the published gap, as shown in Equation (10.10).

$$[\hat{Y}, \hat{I}, \hat{m}, \hat{y}_m, \hat{w}] \leftarrow [\hat{Y}, \hat{I}, \hat{m}, \hat{y}_m, \hat{w}] + L\left(y_m + \varepsilon - \hat{y}_m\right) \tag{10.10}$$

where L is the steady-state Kalman gain of the block $[Y, I, m, y_m, w]$ linearized at potential and discretized over a month, with w a random walk, process weights 10^{-7}, 10^{-8}, 0, 0 and 2×10^{-5}, and a measurement weight of $\max(4 \times 10^{-6}, \sigma_{pub}^2)$. These are teaching values. The gain is $L = (1.439, 0.748, -0.052, 0.608, 1.400)$ in the reference economy and $(0.891, 0.439, 0, 0.464, 0.655)$ in case 4, where the offset is off and the prints are noisy. The private shock w plays the role of the added disturbance state of Section 1.6.3, as a fuel bias does in the offset-free controller of Volume 9, Chapter 4. A perfect-state setting hands the MPC the true states instead, as a labeled comparison.

10.5.6 Eight Budgets as a Quadratic Program

At every budget, and whenever a setting has made its plan stale, the MPC linearizes the economy at $\hat{x}$ and the budget in force by central differences, with the private shock as an eighth state that it expects to fade, $\dot{w} = -w/\tau_w$ with $\tau_w = 2\,\text{yr}$. The linear model, with the affine term the linearization leaves, is converted to quarterly steps, $T_s = 0.25\,\text{yr}$, by an exact zero-order-hold discretization, and the MPC solves the quadratic program (QP) of Equation (10.11) with the framework's linear MPC, `core/linmpc.js`, which Section 1.6.3 describes.

$$\begin{aligned} \min\ & \textstyle\sum_{k=1}^{N} T_s \left[(\hat{Y}_k - 1)^2 + \lambda_B B_k^2 \right] + \nu \textstyle\sum_j (B_j - B_{j-1})^2 \\ & + \rho_d\, \varsigma_d^2 + \rho_b\, \varsigma_b^2 \\ \text{s.t. } & B_{min} \le B_j \le B_{max}, \qquad |B_j - B_{j-1}| \le \Delta_B \\ & r_k b_k - s_k \le 0.03 + \varsigma_d, \qquad b_N \le b_{ceil} + \varsigma_b, \qquad \varsigma \ge 0 \end{aligned} \tag{10.11}$$

where $N = H/T_s = 32$, the budget is held over blocks of four quarters, one per annual budget, B_{-1} is the budget in force, $\lambda_B = 0.2$, $\nu = 2$, the deficit row is the linearized deficit at every quarter, and its slack price ρ_d is 400 for the strict setting, 4 for the soft one and absent when the limit is off, while the ceiling's ρ_b is 2000. The deficit limit is therefore soft in every setting, with a breach priced so high in the strict one that the plan avoids it whenever it can, which is how a limit is written as a cost [5]. On the annual calendar the program has 10 variables, eight budgets and two slacks, and 67 rows. It is solved by the dual active-set method of [18]. The plan's first budget is the request that the calendar then rounds and caps, and if a solve were to fail the budget in force would be kept. Case 8 needs 13 solves, one at the start and one each October, none fails, and the twelve October solves take at most two active-set iterations, 0.25 on average.

10.5.7 Economy, Belief and the Models Behind Them

Equations (10.1) to (10.4), with the shocks, the print errors and the premium, are the simulated economy. The treasury's model is the same equations without knowledge of the private shock, corrected by one published number a month, and linearized with a shock it expects to fade over two years and no foresight of the next one. The models both stand for are the multiplier-accelerator of [11] with Phillips's lags [1] and the estimated reaction functions and premiums of [6, 9, 10], whose coefficients are not copied here. The application estimates no model from data, and the PID, Bohn's rule and the brake carry no model of the economy.

10.6 Ten Cases on One Machine

Table 10.4 lists the cases. Each is loaded from the dropdown, which applies the defaults of Tables 10.2 and 10.3 with the changes its row names, and each run covers the years its row gives, on the default seed. A variant is the case with the named settings changed before Run, and the application gives the same run when the settings are carried through Reset instead, since Reset keeps the reader's settings (Section 1.6.1). A starting debt set in the Model dialog before Run takes effect from the start. Table 10.5 collects the outcomes on the true gap, which the treasury never sees. The quarterly count and the deficit years are those of the scorecard.

Table 10.4 Cases of the Simulator With the Crisis Challenge and the Reference

#	Change from the reference, run length	Teaching point
1	MANUAL, offset off, investment collapse at 0.25 yr, 10 yr	a budget on late figures
2	offset off, demand −2 % for good at 1 yr, 12 yr	stabilizers as droop
3	2 with the PID, K_c 1.5, no integral, monthly, lags 0.02 yr	Phillips 1954
4	2 with the PID, K_c 1.5, τ_I 2 yr, σ_{pub} 0.5 %, 24 yr	Phillips 1957
5	r_0 4 %, no premium, no shock, 30 yr	$r > g$
6	Bohn rule, φ 0.15, b_0 120 %, no shock, 20 yr	the fold
7	brake, investment boom at 1 yr, collapse at 6, 14 yr	a bounded integrator
8	MPC, financial crisis at 2 yr, 12 yr	a procyclical limit
9	pandemic at 1 yr, 12 yr	a decade of debt
C	financial crisis at 2 yr, stabilizers only, 12 yr	three limits at once
R	no shock, 10 yr	the equilibrium

Table 10.5 Lowest and Highest True Gap, Bad Quarters, Deficit Years and Final Debt Over Each Case's Own Run

Case and configuration	y_{min} (yr)	y_{max} (yr)	Qtr	Yr	b_{end}
1, no budget	−6.23 (2.02)	0.97 (5.42)	10	2	95.7
1, K_c 1, no integral	−5.48 (1.83)	4.71 (4.04)	21	2	100.2
1, 2 % at the first October	−3.80 (1.62)	3.83 (5.50)	32	2	94.7
2, as shipped	−5.22 (2.21)	0.00	43	11	112.9
2, s_{tr} 0	−9.90 (2.63)	0.00	39	8	109.7
2, offset on	−4.21 (1.92)	0.45 (3.79)	6	2	94.0
3, P only	−1.28 (1.31)	0.00	0	11	112.2
3, τ_I 2 yr	−1.27 (1.27)	0.00	0	11	111.6
4, as shipped	−5.88 (2.04)	5.08 (4.81)	59	22	148.1
4, K_c 0.5, τ_I 4 yr	−5.15 (2.00)	2.22 (3.98)	17	22	142.5
4, the same and τ_D 0.5 yr	−7.89 (2.15)	4.46 (19.96)	43	22	139.1
5, stabilizers	0.00	0.00	0	30	201.0
5, Bohn ρ_B 0.05	−0.55 (14.38)	0.00	0	30	150.0
5, Bohn ρ_B 0.2	−1.78 (5.50)	0.14 (18.23)	0	7	104.9
6, Bohn ρ_B 0.05	−2.29 (2.44)	0.00	10	20	lost 11.7
6, ρ_B 1, B_{min} −10 %	−6.34 (5.75)	0.00	31	20	lost 15.2
6, stabilizers	0.00	0.00	0	20	lost 10.5
7, brake	−6.08 (7.60)	5.42 (2.40)	25	3	91.0
7, Bohn rule	−5.09 (7.58)	5.79 (2.46)	22	3	90.2
7, stabilizers	−4.96 (7.58)	5.72 (2.46)	23	3	90.3
8, MPC, strict limit	−6.29 (3.69)	1.64 (7.04)	10	1	85.2
8, MPC, soft limit	−4.77 (3.46)	1.27 (5.67)	7	1	96.0
8, PID	−4.36 (3.38)	3.76 (5.17)	11	1	100.4
8, PID, Maastricht clip	−4.97 (3.50)	2.92 (6.35)	14	1	97.2
8, MPC, strict, offset off	−9.30 (4.48)	0.00	21	3	105.3
9, stabilizers	−6.31 (1.92)	5.30 (3.23)	9	1	99.3
9, Bohn rule	−6.36 (1.96)	4.35 (3.25)	9	1	95.8

Gap in % of potential at the year given, debt in %. "Lost" is the year the debt passes 250 %. Qtr: quarters outside ±2 %. Yr: calendar years over a 3 % deficit.

10.6.1 By Hand, and by Nobody

In case 1 the minister holds the budget while investment collapses from year 0.25. Left at zero, the gap bottoms at −6.23 % at year 2.02 and the recovery peaks at +0.97 % at year 5.42, with 10 bad quarters. The recognized gap passes −1 % only at year 0.91, after the first October. The rule of the Introduction, one point of stimulus per point of published gap, sets 0.5 % at that October, 2.5 % at the next, capped, and 2.9 % at the October of year 2, when the published gap reads −2.89 % and the true one −1.46 %. A year later the published gap is +3.09 %, the true one +4.25 %, and the cap lets the budget fall only to 0.9 %. Twenty-one quarters are outside the band, and the economy is still swinging at year 10, at −3.77 %. A minister who asks for 2 % at the first October and leaves it there gets the shallowest trough of the three, −3.80 %, and then a lasting boom with 32 bad quarters, because with the central bank off nothing ever takes the stimulus away.

Table 10.6 The Largest Gain Whose Case-4 Cycle Decays, by How Often the Budget Is Decided

Integral time τ_I	Continuous time	Every month	Each October
none	1.85	1.53	0.76
4 yr	1.55	1.31	0.70
2 yr	1.28	1.11	0.63

Recognition 0.25 yr, implementation 1 yr, no print error, no quantum, a 10 % cap. Engine columns judged on years 32 to 40 against 12 to 20.

Case 2 removes the decision altogether. Private demand falls 2 % for good at year 1, the gap reaches −5.22 % at year 2.21 and settles at −3.02 %, the droop offset of Section 10.4. Without the benefit response the trough is −9.90 % and the accelerator rings. The stabilizers are paid for with a deficit of 4.21 % at year 12 against 1.8 % before the shock, so the debt rises from 90 % to 112.9 % by year 12 and 162.4 % by year 30. With the central bank's offset on, the gap closes to −0.07 % at year 12, only 6 quarters are bad, and the debt ends at 94.0 %.

10.6.2 Phillips 1954 and 1957

Case 3 hands the same shortfall to proportional policy deciding every month with lags of a week. At $K_c = 1.5$ the gap falls to −1.28 % at year 1.31 and approaches the steady-state offset $-2/(0.6625 + 1.5) = -0.925$ %, which the check confirms within 0.003 points after twenty years. At year 12 the gap is −0.90 % and the budget 1.4 %. An integral time of two years removes the offset by year 12 with the budget at 2.0 %, the whole shortfall, which is Phillips's 1954 result. The integral is not free. The deficit stays at 4.41 % and the debt reaches 111.6 % by year 12, because with no central bank, holding output up against a lasting shortfall means paying for it every year.

Case 4 keeps those gains and restores the lags of a real budget, a quarter of recognition, an October calendar and a year of implementation, with a print error of 0.5 %. The loop never settles. The gap swings between −5.88 % and +5.08 %, with peaks at years 9.12, 13.81, 18.10 and 22.12, 4.33 years apart on average, and 59 of 96 quarters are bad, as shown in Figure 10.3. Linearized in continuous time, the same gains are stable without the lags, with a slowest eigenvalue of −0.319 per year, and unstable with them, at $+0.063 \pm 2.545i$, a growing cycle of 2.47 years, as the application's check confirms. The cycle the engine shows is longer, because budgets come once a year, and bounded, because the cap holds it. On the shipped prints 19 of the 24 budgets move the full 2 %. Without the print error 23 of 24 do, the budget cycles through 0, 2, 4 and 2 % with a period of exactly four years, and 58 quarters are bad. Raising the cap to 10 % lets the same loop grow until the budget touches both bounds and the gap swings from −14.76 % to +13.25 %, while a cap of 1 % holds the late swings within ±2.22 % and leaves 15 bad quarters. The per-decision cap is a rate limit that turns an unstable linear loop into a limit cycle whose size it sets.

> **TRY IT IN THE APP** **4 · Phillips 1957: The Same Gains With Lags**
>
> Load the case and press Run. Watch the output gap trend. With Phillips's gains and the lags of a real budget the gap never settles: it swings between −5.88 % and +5.08 %, the scorecard counts 59 of 96 quarters outside the band, and at year 24 the status line reads SPIRAL.

The gain the lags allow depends on how often the budget is decided, as shown in Table 10.6. Section 1.4.11 counts a sample time as about half a period of added dead time, so a budget held for a year adds about half a year of delay to a loop whose lags alone bound the proportional gain at 1.85. The continuous-time column is the boundary of the linearized loop, found by bisection on its eigenvalues. For proportional policy it is the ultimate gain of Section 1.4.7. The other two are found on the engine, with the print error and the quantum removed and the cap raised to 10 % so that none of them shapes the answer, by bisection on whether the gap's swing over years 32 to 40 is smaller than over years 12 to 20. Monthly decisions cost 13 to 17 percent of the continuous-time gain, and

an annual budget allows half to 57 percent of what monthly decisions allow, 0.76 against 1.53 for proportional policy and 0.63 against 1.11 with an integral time of two years. Case 4's gain of 1.5 is more than twice the annual limit, and the weaker policy of the case's second step, $K_c = 0.5$ with $\tau_I = 4\,\text{yr}$, lies inside it. That policy leaves 17 bad quarters, seven in the first slump and the rest in smaller swings that end by year 10.

🕮 **Pause.** Try Exercise 10.2 before reading on.

The derivative term of the third step helps only on clean data. On the noisy prints, $\tau_D = 0.5\,\text{yr}$ raises the bad quarters from 17 to 43 and the gap's RMS after year 6 from 1.03 % to 2.35 %, because each quarter's new print error becomes a jump in the measured rate and a budget swing at the next October. On seeds 1 to 10 the same two settings leave 9 to 19 and 23 to 46 bad quarters, so the ordering does not depend on the draw. Without the print error the same term lowers the RMS from 0.80 % to 0.42 % and the bad quarters from 11 to 8, since it recovers phase that the exponential lags take, and the linearized loop agrees, its slowest eigenvalue moving from -0.112 to -0.115. The expectation of the application's original design, that the derivative term would make the lagged loop worse, holds only through the data.

🕮 **Pause.** Try Exercise 10.3 before reading on.

10.6.3 When r Exceeds g, and Past the Fold

Case 5 changes nothing in the economy except the borrowing rate, now 4 % against growth of 2 %, with no premium and no shock. Output stays at potential, and the reference budget's primary deficit of 0.9 % now drives the debt from 90 % to 119.9 % at year 10, 156.4 % at 20, 201.0 % at 30 and 255.4 % at 40, the closed form $b(t) = (b_0 + \delta/(r-g))\,e^{(r-g)t} - \delta/(r-g)$ with $\delta = 0.009$ to within 10^{-13} at year 30. Holding 90 % still would take a surplus of 1.8 %. Bohn's rule at its shipped 0.05, above $r - g = 0.02$, holds the debt but far above its target, 150.0 % at year 30, 159.4 % at 40 and 175.5 % at 80, on its way to the proportional controller's equilibrium $(\delta + \rho_B b^*)/(\rho_B - (r-g)) = 180\,\%$. At 0.2 the debt settles at 104.9 %, against the predicted 105 %, with a primary surplus of 2.1 %. A proportional rule on the debt leaves an offset, as the stabilizers leave one on the gap.

Case 6 starts at 120 % with a premium of 0.15 points per point above 100 %, past the fold of 100.71 %. The rate is 5.62 % by year 2, reaches its cap of 11 % by year 8, and the shipped rule, cutting the budget by up to 5 %, loses market access at year 11.69. A coefficient of 1 with cuts of up to 10 % delays the loss to year 15.17 and takes the economy into a slump of −6.34 % on the way, and the stabilizers alone lose access at year 10.54. The application's sweep of coefficients 0.05, 0.2 and 1 against cut limits of 2, 5 and 10 % loses access in all nine runs, between years 11.3 and 15.2. This is the honest case. A cut shrinks output before it improves the balance, the smaller output raises the ratio, and the ratio raises the rate. From 102 % the shipped rule brings the debt back to 92.2 % in thirty years, and bisection on the engine places the highest start from which it recovers at 103.47 %, so the rule moves the threshold up by less than three points from the fold of the fixed budget. On the Model dialog's steps of one point the shipped rule recovers from 103 % and not from 104 %. Volume 5, Chapter 9 meets the same limit of authority in a pension fund whose deficit grows on its own while a capped contribution refills it.

> **TRY IT IN THE APP** **6 · The Fold: A Spiral No Surplus Stops**
>
> Load the case and run it. The debt starts at 120 %, past the fold, and Bohn's rule cuts the budget while the borrowing rate climbs to its cap of 11 % by year 8. Market access is lost at year 11.69. Then choose the stabilizers alone as the controller, press Reset and run again: access is lost sooner, at year 10.54.

10.6.4 Rules and an Optimizer Through a Crisis

Case 7 runs a boom from year 1 and a collapse from year 6 under the brake, whose account fills to +2.00 % at the October budget of year 2 as it cuts into the boom, and empties to −1.10 % at the

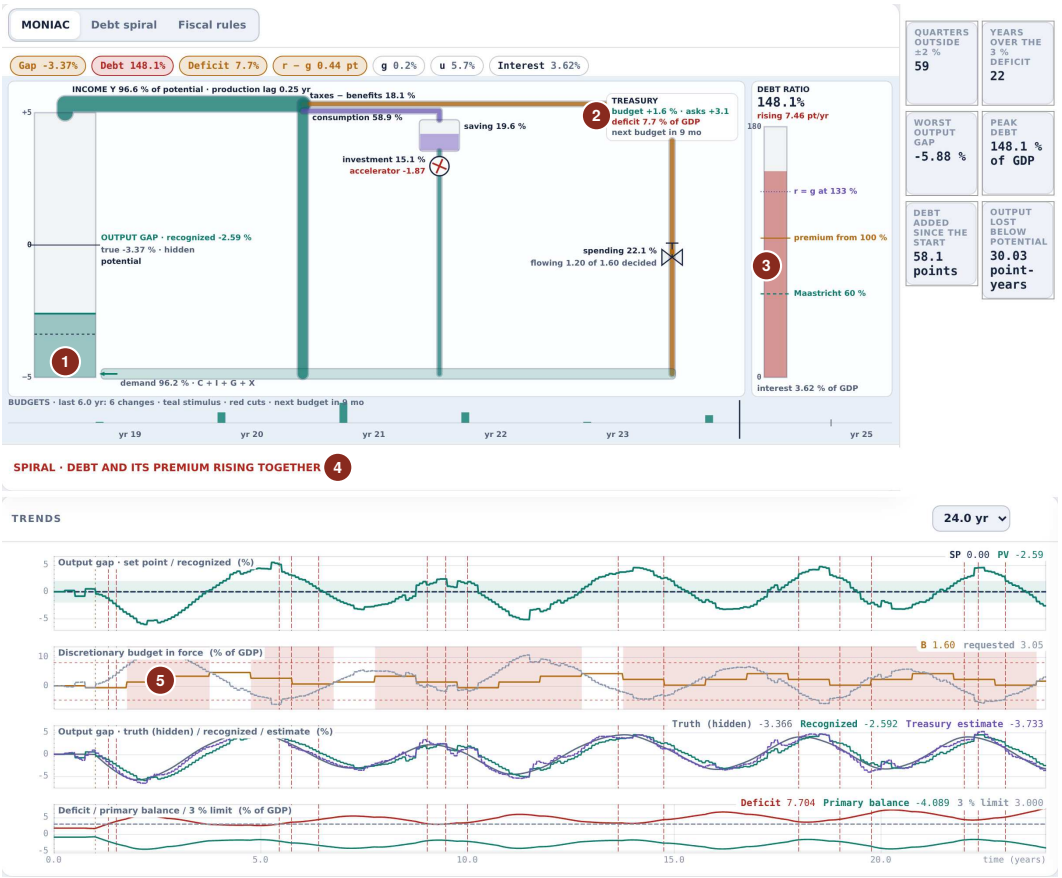


Figure 10.3 Case 4, Phillips's 1954 gains under the lags of 1957, at $t = 24.00$ yr in AUTO in the MONIAC view with a 24-year trend window, as loaded, a composite of the plant view, the trends and six scorecard tiles set in gray frames at the upper right. The badges read a gap of −3.37 %, debt 148.1 %, deficit 7.7 %, $r - g$ of 0.44 points, growth 0.2 %, unemployment 5.7 % and interest 3.62 % of output. The output tank shows the published gap of −2.59 % as its surface and the hidden true gap of −3.37 % as a dashed line, income is 96.6 % of potential, consumption 58.9 %, saving 19.6 %, taxes less benefits 18.1 %, investment 15.1 % with the accelerator taking 1.87 %, and spending 22.1 %, with 1.20 of the 1.60 points decided flowing. The treasury holds a budget of +1.6 % and its PID asks for +3.1 %, with the next budget in nine months. The debt column reads 148.1 %, rising 7.46 points a year past the premium threshold of 100 % and the level of 133 % at which r would equal g, and the status strip reads SPIRAL. The budget strip counts six changes in six years, each a full two points except the last. In the trends the published, true and estimated gaps cycle together, the budget moves in annual steps with the cap's shading over 19.0 of the 24 years by the run's record, and the deficit climbs past 7 %. The scorecard counts 59 quarters outside ±2 %, 22 deficit years, a worst gap of −5.88 %, a peak debt of 148.1 %, 58.1 points of debt added and 30.03 point-years of output lost. Callouts: (1) the output tank with the true gap dashed below the published surface, (2) the treasury's budget and request, (3) the debt column with its premium threshold, (4) the status line, (5) the budget in force with the cap's shading.

October of year 7 as it stimulates into the bust. BRAKE BINDING is first raised by the monthly run at year 7.83, the one after that budget, and the debt ends at 91.0 %. The cycle is no smaller. The brake leaves 25 bad quarters against 23 with the stabilizers alone and 22 under Bohn's rule, whose

budget stays between −0.3 % and +0.4 %. Its stimulus of 1.2 % decided at the October of year 4 was withdrawn at the October of year 5, so the withdrawal was still reaching demand as investment fell, and the bust's trough is −6.08 % against −4.96 % with no discretion. Neither rule can time a cycle through an annual budget, which is the lesson of case 4 in another form.

> ✦ DID YOU KNOW? **A Brake That Spends Into the Bust**
>
> From the simulator: in case 7 the brake leaves 25 bad quarters and a trough of −6.08 %, against 23 and −4.96 % with the stabilizers alone (Table 10.5). A stimulus decided one October and withdrawn the next was still being withdrawn when investment fell.

Case 8 is the challenge's financial crisis, a collapse and a rescue of 10 % of output at year 2, under the MPC. With the strict deficit limit the plan cuts the budget to −1.9 % at the October of year 2, when the stabilizers have already raised the deficit to 3.32 %, and to −3.9 % a year later, at a deficit of 4.75 %, a full cap's move in the middle of the slump. Both plans are bound by the deficit row of their first quarter, and even the strict price leaves it 0.14 and 0.89 points above the limit. The trough deepens to −6.29 % at year 3.69 and 10 quarters are bad, for the lowest final debt of the case, 85.2 %. With the soft limit the plan never cuts below −0.9 %, the trough is −4.77 % and 7 quarters are bad, with a debt of 96.0 %, better on both counts than the PID at its defaults, which leaves 11 bad quarters, a boom of +3.76 % and a debt of 100.4 %. The PID under the Maastricht clip is worse still, at 14 bad quarters, because the clip also cuts in the slump. With the central bank off, the strict-limit MPC drives the gap to −9.30 %. An optimizer does what its constraints say, and a deficit limit that binds in a slump is procyclical whoever enforces it.

Case 9 is a pandemic, a year of lockdown that removes 10.5 % of private demand while emergency transfers of 10 % are paid. The aid enters consumption at $c = 0.75$, offsetting 7.5 of the 10.5 points, and the rest takes the gap to −6.31 % at year 1.92 and the deficit to 16.7 %. The debt jumps from 90 % to 109.2 % by year 2, falls to 97.6 % at year 3 as output rebounds to +5.30 %, and is still 99.3 % at year 12. At $r - g = -1$ point a one-off debt fades by about one percent of itself a year, so nine points remain a decade later, or six under Bohn's rule.

10.6.5 The Crisis Challenge

The challenge repeats the crisis of case 8 with the stabilizers alone and asks for no more than six quarters outside ±2 % in twelve years, no more than two calendar years with a deficit above 3 %, and a debt at or below 100 % at year 12, judged on the truth. Table 10.7 compares nine configurations, and Figure 10.4 draws its bad quarters and final debts against the two limits. As shipped the gap falls to −4.77 % at year 3.46, total investment from 17.0 % to 13.3 % of output at year 3.08, and the debt peaks at 107.3 % at year 3.71, the rescue plus a smaller output, before ending at 99.5 %. Nine quarters are bad, six in the slump and three in the boom that follows, to +2.31 % at year 5.62, as the central bank's offset overshoots. The deficit limit and the debt target are met, so the stabilizers fail on the gap alone.

The suggested setting raises the tax stabilizer to 0.3, which acts at once, and adds the brake at 0.5 per point of gap, which leans against the slump at the Octobers of years 2 and 3 with 0.6 % and 0.9 % and then against the boom with cuts down to −1.1 %. It leaves five bad quarters, all in the slump, a trough of −4.14 %, a boom held to +1.88 %, one deficit year and a debt of 98.9 %, with the account between −1.50 % and +1.30 % against its band of ±1.55 %. Halving the step changes its final debt by 1.0×10^{-5} and neither verdict. The tax rate alone meets the challenge exactly at its limit of six quarters, and the brake alone does not, at seven. A brake gain of 1 with the same tax rate also meets it, with the smallest swings of the table, but over 24 years its cycle grows, to a swing between −1.82 % and +1.71 % in the last six years, while at 0.5 the swing has decayed to between −0.14 % and +0.37 %. Twelve years are too short to show that instability, which is case 4's lesson at a smaller gain. The original design proposed a debt target of 90 %. Of 68 settings of the MPC, Bohn's rule and

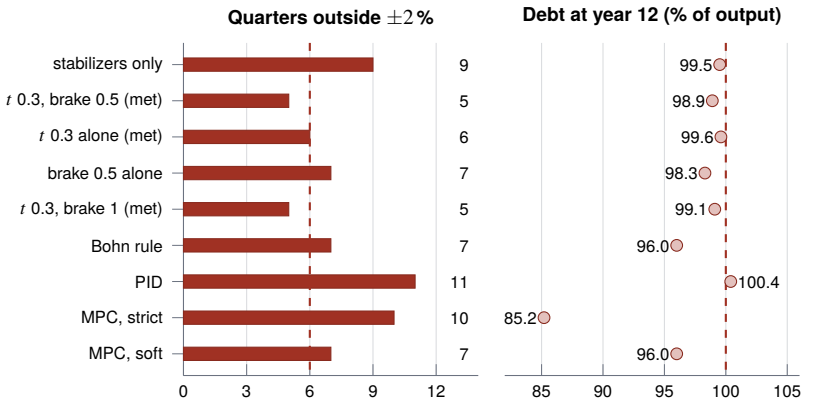


Figure 10.4 Two columns of Table 10.7, the crisis challenge over twelve years: the quarters with the true gap outside ±2 % (bars, values at the right) against the limit of six, and the debt at year 12 (dots) against the limit of 100 % (dashed lines). Notice that every configuration meets the debt limit except the PID, so the challenge is lost on the gap, and that the configuration with the lowest debt, the MPC with the strict deficit limit, has the second most bad quarters.

Table 10.7 The Crisis Challenge of Twelve Years in Nine Configurations

Configuration	Qtr	Yr	b_{12}	y_{min}, y_{max}	Target
stabilizers only, as shipped	9	1	99.5	−4.77, 2.31	not met
t 0.3 and brake K_{br} 0.5, suggested	5	1	98.9	−4.14, 1.88	met
t 0.3 alone	6	1	99.6	−4.48, 1.91	met
brake K_{br} 0.5 alone	7	1	98.3	−4.36, 2.13	not met
t 0.3 and brake K_{br} 1	5	1	99.1	−3.82, 1.30	met
Bohn rule, ρ_B 0.05	7	1	96.0	−5.19, 1.93	not met
PID, K_c 0.5, τ_I 8 yr	11	1	100.4	−4.36, 3.76	not met
MPC, strict deficit limit	10	1	85.2	−6.29, 1.64	not met
MPC, soft deficit limit	7	1	96.0	−4.77, 1.27	not met

Limits 6 quarters, 2 years and 100 % of output. Gap in %.

the brake measured on this case, none reaches it within the gap and deficit limits, and the lowest debt among them, 67.1 % under a Bohn coefficient of 0.5 with a target of 80 % and a tax rate of 0.3, comes with 24 bad quarters. Repaying a rescue of 10 % in a decade at $r - g = -1$ point needs surpluses that the gap band does not allow.

10.7 Technology in Practice

The treasury of Section 10.5 reads a published gap and an exact debt, and acts through a budget that passes a calendar, a quantum and a cap. Figure 10.5 draws the loop with the institutions of a real budget in place of the schematic's symbols.

What Measures It

Output is compiled, not measured by a device. National accounts are assembled from surveys and administrative records, published first about a month after a quarter ends and revised several times after that [19]. The gap needs potential output as well, a trend whose end-of-sample estimate is the larger source of error in real time [20]. In the model the recognition lag τ_{rec} of 0.25 yr and the quarterly print error σ_{pub} of Equation (10.3) stand for the late print and its revisions, and the debt is read exactly, as a treasury reads its own accounts.

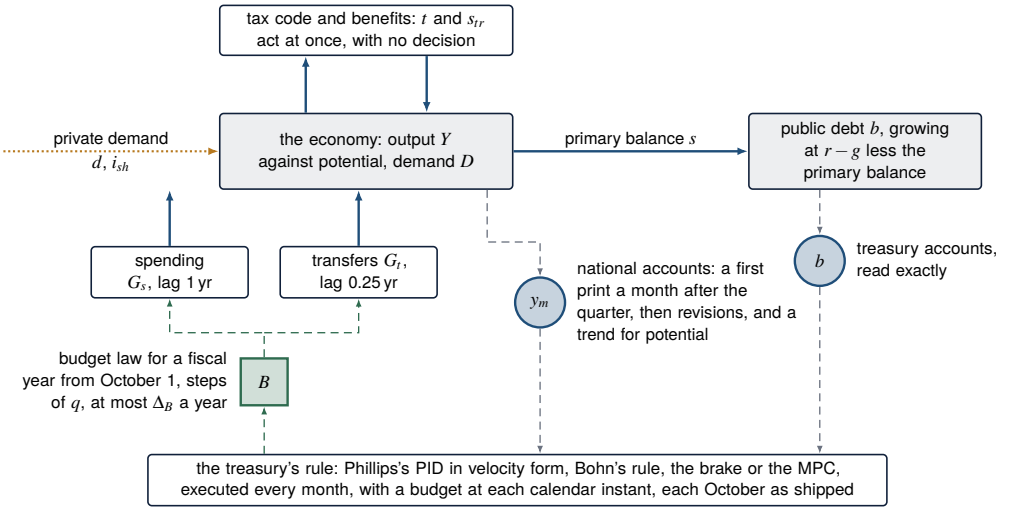


Figure 10.5 The budget loop with the institutions of a real budget in place of the schematic's symbols. National accounts and an estimate of potential output give the published gap y_m, the treasury's accounts give the debt b, and a rule sets the budget B, which reaches demand through a spending and a transfer pipeline. Notice the inner loop that needs no decision: the tax code and the benefits answer output at once, while the treasury acts once a year through two lags.

TECHNOLOGY Sensor: Output Read From Revised Accounts

No instrument reads output. Statistical agencies aggregate surveys and administrative records into national accounts. In the United States the first, or advance, estimate of a quarter's output appears about one month after the quarter ends, a second and a third estimate follow in the next months, and comprehensive revisions come later [19]. The gap also needs potential output, a trend. Revisions of real-time gap estimates have been found to be of the same order of magnitude as the gap itself and highly persistent, and the real-time estimates were severely biased around turning points of the cycle [20]. On a trend the failure is a published gap that later moves, most near a turning point.

What Moves It

The final element is the budget law. The model's annual calendar instant, each October, is the start of the fiscal year of the United States Treasury [21]. The quantum q and the cap Δ_B of Equation (10.7) make each budget a step of tenths of a point of output, at most two points from the last. The marginal tax rate t and the benefit response s_{tr} are the automatic stabilizers, which act without a decision.

TECHNOLOGY Final Element: A Budget Adopted Once a Year

The fiscal year of the United States Treasury begins on October 1 and ends on September 30 [21], so a discretionary budget is set once a year. The tax code acts without a decision. The federal income tax reduces the multiplier effects of demand shocks through the marginal taxation of income fluctuations, and such automatic stabilizers avoid the slow implementation that can make discretionary policy lag far behind events. Unemployment insurance answers the cycle too, but unemployment lags output, and the stabilizing effect of the taxes appeared to be several times that of the benefits [22]. Its failure is lateness, a stimulus that lands in the recovery, as in case 1.

Where the Controller Runs

No computer closes this loop. The budget is adopted for a fiscal year [21], so the decision period is one year, twelve times the control period T_c of one month at which the simulated treasury executes. A

central bank decides more often: the Federal Open Market Committee holds eight regularly scheduled meetings a year [23], between the monthly and the quarterly calendars of the app. In the simulator the PID of Equation (10.8) is the framework's velocity-form block, executed every month and allowed to change the budget only at a calendar instant, and Bohn's rule, the brake and the MPC take its place in the same slot.

> **IN PRACTICE A Ceiling Computed From the Cycle**
>
> The Swiss debt brake turns the cycle into a spending limit with one formula. Maximum expenditure equals revenue times a factor k, the ratio of real trend output to actual output, so a deficit is allowed when output is below trend and a surplus is required above it. Trend output comes from a modified Hodrick-Prescott filter on 24 annual points with a smoothness parameter of 100, and revenue is assumed to move one for one with output. The parameters were kept simple and transparent on purpose, so that the ceiling would not become a black box for policymakers and voters [15].

What Is Modernizing It

Two developments shorten the recognition lag. The first is nowcasting. A dynamic factor model cast in state-space form reads many series as they are released, and a Kalman filter updates its estimate of current growth with each release, weighting each by its timeliness and quality. One central bank's staff nowcast is updated daily when new data arrive and published every week [19]. The treasury's estimate of Equation (10.10) is the same kind of filter fed by one published number a month. The second is the estimate of potential output itself. Estimates made by central bank staff after the mid-1990s had considerably better revision properties than earlier ones, circumstantial evidence that the procedures for estimating potential output improved [24]. The predictive controller of Section 10.5 shows what a model-based plan adds to either estimate, and case 8 shows what its constraints take away.

10.8 Limitations

The economy is a teaching model with backward-looking expectations. Households spend out of current disposable income without anticipating the taxes that a debt implies, firms invest on current growth, and lenders price today's debt. A rule that the public believed, a brake or a debt target, would change how households save, how firms invest and what premium lenders charge. Every case is therefore exposed to the Lucas critique [13], and it measures a rule acting on households and lenders who never learn it, not on a public that anticipates it. The application has no inflation and so no anchored share of expectations, which the central bank of Chapter 11 carries.

Output is set by demand and potential grows at a fixed rate, the central bank's offset is an integral standing in for monetary policy, and the borrowing rate ignores it. The economy is real and closed, with no maturity structure, so the whole debt reprices at once. The premium is a fixed function of the debt, the fold is a static threshold rather than a run on the debt, and the market-access limit of 250 % is a convention. The coefficients, lags and thresholds are teaching choices, so the figures reported here describe no particular country.

The treasury has one discretionary instrument split between spending and transfers, with no tax-rate decisions. The observer has fixed teaching weights and expects every private shock to fade over two years, the MPC knows nothing of shocks to come, and its deficit limit is a priced soft constraint even in the setting labeled strict. The brake is simplified, with the stabilizers allowed in full, the bound enforced at every budget rather than corrected over several years, and credits kept as debt repaid, and the Maastricht rule clips only the deficit a budget plans at today's revenue.

OPEN CHALLENGE **Where Potential Output Is Today**

Every rule that leans on the gap needs today's potential output, the end of a trend that no one can yet see. A study of real-time estimates found that the bulk of their revisions came not from revised data but from the pervasive unreliability of end-of-sample estimates of the trend [20]. Estimates made by central bank staff after the mid-1990s held up considerably better in real time, an improvement that purely statistical detrending methods did not share [24]. Progress would be estimates of potential that hold when the budget is decided, not only after the revisions.

10.9 Conclusion

This chapter studied a simulator that joins Phillips's multiplier-accelerator, drawn as his hydraulic machine, to a treasury that decides once a year through a recognition lag, a quantum and a cap, and to a debt ratio whose borrowing rate rises past a threshold. The stabilizers cut the multiplier from 4 to 1.51 and leave an offset of −3.02 % after a lasting fall of 2 % in demand. Integral policy removes it at the cost of a debt that reaches 111.6 % in twelve years, the lags turn Phillips's 1954 gains into a cycle with 59 of 96 bad quarters, and an annual budget allows a proportional gain of 0.76 where monthly decisions allow 1.53. Above $r = g$ a debt grows from 90 % to 201 % in thirty years with no shock, and from 120 %, past a fold of 100.71 %, no rule in the sweep avoids losing market access. In the crisis challenge the stabilizers alone leave nine bad quarters and a stronger tax with the brake leaves five. Expectations look backward and every coefficient is a teaching value. A fuller model would add a public that learns the rule, a maturity structure for the debt, inflation, and a premium that responds to expectations of default rather than to the debt alone. The application makes the lag, the calendar and the debt of fiscal policy measurable on one machine.

KEY POINTS

- Taxes that fall and benefits that rise in a slump steady the economy at once, but alone they leave a gap of −3.02 % after a lasting fall of 2 % in demand.
- Integral policy removes that offset at the cost of a debt that reaches 111.6 % in twelve years.
- An annual budget allows a proportional gain of 0.76 where monthly decisions allow 1.53, and the lags turn Phillips's 1954 gains into a cycle with 59 of 96 bad quarters.
- Above $r = g$ a debt grows from 90 % to 201 % in thirty years with no shock, and past the fold of 100.71 % no rule in the sweep avoids losing market access.

Where else this applies. Volume 5, Chapter 9 refills a fund through a capped contribution decided a year late, against a deficit that grows on its own.

References

[1] A. W. Phillips, "Stabilisation policy in a closed economy," *The Economic Journal*, vol. 64, no. 254, pp. 290–323, 1954.

[2] ——, "Stabilisation policy and the time-forms of lagged responses," *The Economic Journal*, vol. 67, no. 266, pp. 265–277, 1957.

[3] ——, "Mechanical models in economic dynamics," *Economica*, vol. 17, no. 67, pp. 283–305, 1950.

[4] C. Bissell, "Historical perspectives – The Moniac A hydromechanical analog computer of the 1950s," *IEEE Control Systems*, vol. 27, no. 1, pp. 69–74, 2007.

[5] R. S. Pindyck, "Optimal policies for economic stabilization," *Econometrica*, vol. 41, no. 3, pp. 529–560, 1973.

[6] H. Bohn, "The behavior of U. S. public debt and deficits," *The Quarterly Journal of Economics*, vol. 113, no. 3, pp. 949–963, 1998.

[7] E. M. Leeper, "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, vol. 27, no. 1, pp. 129–147, 1991.

[8] D. E. Seborg, T. F. Edgar, D. A. Mellichamp, and F. J. Doyle, III, *Process Dynamics and Control*, 4th ed. Hoboken, NJ: John Wiley & Sons, 2016.

[9] T. Laubach, "New evidence on the interest rate effects of budget deficits and debt," *Journal of the European Economic Association*, vol. 7, no. 4, pp. 858–885, 2009.

[10] A. R. Ghosh, J. I. Kim, E. G. Mendoza, J. D. Ostry, and M. S. Qureshi, "Fiscal fatigue, fiscal space and debt sustainability in advanced economies," *The Economic Journal*, vol. 123, no. 566, pp. F4–F30, 2013.

[11] P. A. Samuelson, "Interactions between the multiplier analysis and the principle of acceleration," *The Review of Economics and Statistics*, vol. 21, no. 2, 1939.

[12] N. Barr, "The history of the Phillips Machine," in *A. W. H. Phillips: Collected Works in Contemporary Perspective*. Cambridge University Press, 2000, pp. 89–114.

[13] R. E. Lucas, "Econometric policy evaluation: A critique," *Carnegie-Rochester Conference Series on Public Policy*, vol. 1, pp. 19–46, 1976.

[14] A. J. Auerbach and D. Feenberg, "The significance of federal taxes as automatic stabilizers," *Journal of Economic Perspectives*, vol. 14, no. 3, pp. 37–56, 2000.

[15] T. Beljean and A. Geier, "The Swiss debt brake — has it been a success?" *Swiss Journal of Economics and Statistics*, vol. 149, no. 2, pp. 115–135, 2013.

[16] O. Blanchard, "Public debt and low interest rates," *American Economic Review*, vol. 109, no. 4, pp. 1197–1229, 2019.

[17] O. J. Blanchard, "Public debt and low interest rates," National Bureau of Economic Research, Working Paper 25621, 2019. [Online]. Available: https://www.nber.org/papers/w25621

[18] D. Goldfarb and A. Idnani, "A numerically stable dual method for solving strictly convex quadratic programs," *Mathematical Programming*, vol. 27, no. 1, pp. 1–33, 1983.

[19] B. Bok, D. Caratelli, D. Giannone, A. Sbordone, and A. Tambalotti, "Macroeconomic nowcasting and forecasting with big data," Federal Reserve Bank of New York, Staff Report 830, 2017. [Online]. Available: https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr830.pdf

[20] A. Orphanides and S. van Norden, "The reliability of output gap estimates in real time," Board of Governors of the Federal Reserve System, Finance and Economics Discussion Series 1999-38, 1999. [Online]. Available: https://www.federalreserve.gov/econres/feds/the-reliability-of-output-gap-estimates-in-real-time.htm

[21] "31 U.S.C. 1102, fiscal year," United States Code, 2023 edition, Title 31, Chapter 11, U.S. Government Publishing Office, 2023. [Online]. Available: https://www.govinfo.gov/content/pkg/USCODE-2023-title31/html/USCODE-2023-title31-subtitleII-chap11-sec1102.htm

[22] A. J. Auerbach and D. Feenberg, "The significance of federal taxes as automatic stabilizers," National Bureau of Economic Research, Working Paper 7662, 2000. [Online]. Available: https://www.nber.org/papers/w7662

[23] "Meeting calendars, statements, and minutes (2021–2027)," Board of Governors of the Federal Reserve System, Federal Open Market Committee, 2026, accessed 2026-11-20. [Online]. Available: https://www.federalreserve.gov/monetarypolicy/fomccalendars.htm

[24] R. M. Edge and J. B. Rudd, "Real-time properties of the Federal Reserve's output gap," Board of Governors of the Federal Reserve System, Finance and Economics Discussion Series 2012-86, 2012. [Online]. Available: https://www.federalreserve.gov/pubs/feds/2012/201286/201286pap.pdf

10.10 Exercises

10.1 Concept check · 2 min

In case 2 private demand falls by 2 % of output for good at year 1. The central bank's offset is off, and no budget is decided, so only the taxes and benefits respond. Where does the output gap settle?

(a) At zero, because the benefits keep rising until output is back at potential

(b) Near −3.0 %, the shortfall times the multiplier that the stabilizers leave

(c) Near −8 %, the shortfall times the multiplier $1/(1-c)$ of consumption

(d) It never settles, because the accelerator keeps the economy cycling on its own

10.2 Concept check · 2 min

Table 10.6 shows that, with the same recognition and implementation lags, the largest proportional gain whose cycle decays is 1.53 when the budget is decided every month and 0.76 when it is decided each October. Which explanation is right?

(a) The quantum of 0.1 % rounds away the small corrections that a high gain makes
(b) The cap of two points per decision stops a high gain from acting fast enough
(c) A budget held for a year adds about half a year of delay, which takes phase from the loop
(d) A print error drawn once a quarter is averaged better by twelve decisions than by one

10.3 Concept check · 2 min

In case 4 the weaker policy, $K_c = 0.5$ with $\tau_I = 4\,\text{yr}$, leaves 17 bad quarters on the published data. Adding $\tau_D = 0.5\,\text{yr}$ raises the count to 43. With the print error removed, the same derivative term lowers the count from 11 to 8. Which explanation fits both results?

(a) The derivative is computed only at each October budget, so its rate estimate is a year old
(b) The derivative acts on the error, so every budget decision gives the output a kick
(c) The derivative term replaces part of the integral action that the slow loop needs
(d) Each new quarterly print error is a jump in the measured rate, which the next budget follows

10.4 Theory and calculation · 6 min

The reference economy has $c = 0.75$, a marginal tax rate $t = 0.25$ and a benefit response $s_{tr} = 0.3$. Private demand falls by 2 % of output for good, and the central bank's offset is off.

(a) Find the steady multiplier from spending to output and the gap at which the economy settles with the stabilizers alone. *Checkpoint:* 1.509 and −3.02 %.

(b) Repeat with a marginal tax rate of 0.4, and with no benefit response at the shipped tax rate. *Checkpoint:* 1.290 and −2.58 %, then 2.286 and −4.57 %.

(c) Proportional policy with $K_c = 1.5$ adds a budget of K_c times the negative gap, with no lags, at the shipped stabilizers. Find the steady gap and the budget that holds it. *Checkpoint:* −0.925 % and 1.39 %.

10.5 Theory and calculation · 6 min

Case 5 raises the borrowing rate to $r = 4\,\%$ against growth of $g = 2\,\%$, with no premium and output at potential. The reference budget runs a primary deficit $\delta = 0.9\,\%$ of output, and the debt starts at $b_0 = 90\,\%$.

(a) Solve $\dot{b} = (r - g)\,b + \delta$ for the debt at years 10 and 30, and find the primary surplus that would hold the debt at 90 %. *Checkpoint:* 119.9 %, 201.0 % and 1.8 %.

(b) Bohn's rule sets the budget to $-\rho_B(b - b^*)$ with $b^* = 90\,\%$. Find the steady debt for $\rho_B = 0.05$ and for $\rho_B = 0.2$, and the primary surplus in the second case. *Checkpoint:* 180 %, then 105 % with a surplus of 2.1 %.

(c) State the condition on ρ_B for the rule to hold the debt at all. *Checkpoint:* $\rho_B > r - g = 0.02$.

10.6 Theory and calculation · 6 min

Above the threshold $b_{thr} = 100\,\%$ the borrowing rate carries a premium $\varphi\,(b - b_{thr})$, with $r_0 = 1\,\%$ and $g^* = 2\,\%$. The reference budget holds a primary balance of $s = -0.9\,\%$.

(a) Show that $b = 90\,\%$ is an equilibrium of this budget and that it is stable. *Checkpoint:* $h(0.9) = -0.009$ and $h' = -0.01$ per year.

(b) With the shipped slope $\varphi = 0.03$, find the second equilibrium and the debt's eigenvalue there. *Checkpoint:* 104.67 % and +0.023 per year.

(c) With the case-6 slope $\varphi = 0.15$, find the fold, and say what a debt started at 102 % does under the fixed budget. *Checkpoint:* 100.71 %. It grows.

10.7 App challenge · 8 min

Load scenario 1, The budget by hand. Investment collapses at year 0.25, and the central bank's offset is off. Use the default seed and run each case for 10 yr. For each run read the tiles Quarters outside ±2 % and Worst output gap, and find in the exported file the highest true gap, column `gap_true_pct`, and its time.

(a) Leave the mode at Manual and the Discretionary budget request at 0. *Checkpoint:* 10 quarters, −6.23 %, highest 0.97 % at year 5.42.

(b) Press the reload button. Before Run, switch to Auto, set Controller to Phillips PID on the gap, PID proportional gain to 1 and Integral time to 0. *Checkpoint:* 21 quarters, −5.48 %, highest 4.71 % at year 4.04.

Accept a gap within 0.02 % and a time within 0.05 yr. Say when the stimulus does most of its work.

10.8 App challenge · 8 min

Load scenario 4, Phillips 1957: the same gains with lags. The scenario ships with $K_c = 1.5$, $\tau_I = 2$ yr, a print error of 0.5 % and the annual calendar. Use the default seed and run each case for 24 yr, changing the settings before Run. Read the tiles Quarters outside ±2 % and Worst output gap.

(a) Run the scenario as shipped. *Checkpoint:* 59 quarters and −5.88 %.

(b) Set PID proportional gain to 0.5 and Integral time to 4. *Checkpoint:* 17 quarters and −5.15 %.

(c) Keep (b) and set Derivative time to 0.5. *Checkpoint:* 43 quarters and −7.89 %.

The counts repeat exactly at the default seed. Accept a gap within 0.02 %. Compare the gains of (a) and (b) with Table 10.6.

10.9 App challenge · 8 min

Load the challenge, A financial crisis. Investment collapses at year 2 and a bank rescue adds 10 % of output to the debt. The target is at most six quarters outside ±2 %, at most two years over a 3 % deficit and a debt at or below 100 % at year 12. Use the default seed, make each change before Run and run 12 yr. Read the two count tiles and the debt at year 12, column `debt_pct`.

(a) Run the challenge as shipped, with the stabilizers only. *Checkpoint:* 9 quarters, 1 year, 99.5 %.

(b) Set Controller to Swiss debt brake and Brake stimulus per point of gap to 0.5. *Checkpoint:* 7 quarters, 1 year, 98.3 %.

(c) Keep (b) and set Marginal tax rate to 0.30. *Checkpoint:* 5 quarters, 1 year, 98.9 %.

Accept a debt within 0.1 %. Say which runs meet the target and why the tax rate helps where the brake cannot.

10.10 Application area · 3 min

At the reference equilibrium the overall deficit is 1.8 % of output, below a limit of 3 %. Each point of negative gap adds $t + s_{tr} = 0.55$ points to the primary deficit through the stabilizers.

(a) Find, to first order, the gap at which the stabilizers alone take the deficit to the limit. *Checkpoint:* About −2.2 %.

(b) The gap band of the scorecard is ±2 %. Say what this means for a deficit limit enforced in a slump, and how case 8 shows it.

10.11 Application area · 4 min

A bank rescue adds 10 points of output to the debt, and afterwards the primary balance returns to the level that holds the debt still. Take $r - g = -1$ point, as in the reference economy.

(a) Find how much of the rescue is left in the debt ratio after ten years. *Checkpoint:* About 9.05 points.

(b) Find the constant extra primary surplus that would remove the whole rescue in ten years. *Checkpoint:* About 0.95 % of output a year.

Name one feature of real public debt that the model omits and that would change these numbers.

10.12 Review the agent · 5 min

An AI assistant is asked to retune the Phillips PID of case 4 and replies as follows. “The cycle comes from too much gain. With an integral time of two years the stability limit is 1.28 in continuous time and 1.11 when the budget is decided every month. Keep $\tau_I = 2\,\text{yr}$ and lower K_c from 1.5 to 1.0, which leaves a 10 % margin below the monthly limit, and the cycle will decay.” Find the flaw in the proposal and say how it could be checked.