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CHAPTER 2

Gravity-Drained Tank Laboratory for Level Loops With a Moving Time Constant

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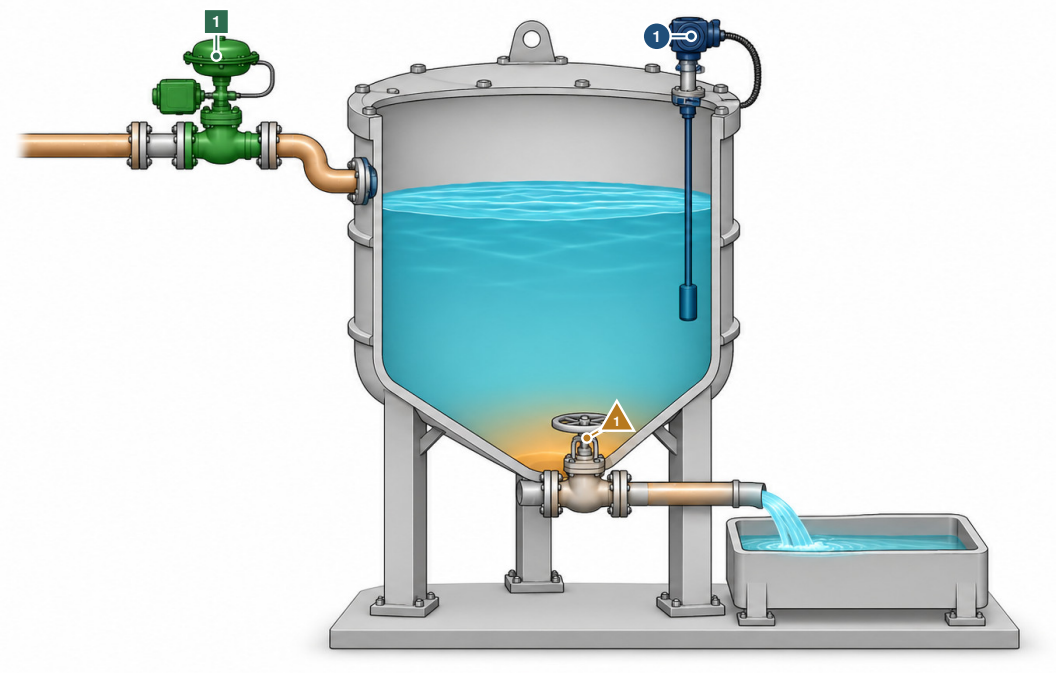
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CHAPTER 2

Gravity-Drained Tank Laboratory for Level Loops With a Moving Time Constant



The controller moves 1 the inlet valve on the supply line

The controller reads 1 the level sensor through the roof

What disturbs it 1 the hand valve on the drain

Notice that the drain sits under the full depth of water, so a fuller tank pushes water out faster, although the model's tank has straight sides and a flat bottom rather than a cone. Illustration, not to scale.

IN PLAIN TERMS **A Tank That Drains Under Its Own Weight**

The main idea. Water flows into a tank through a valve that a controller moves. It drains out through a hand valve at the bottom, and deeper water pushes out faster, so the tank settles at its own level for each inflow.

The challenge. A fuller tank reacts more strongly and more slowly to the same valve move, so settings tested at one depth behave differently at another. With the drain shut, the controller can only fill the tank and never lower it.

Why it matters. A controller that misjudges its tank can overfill it or pass a flow change on to the next unit.

THREE WAYS TO READ THIS CHAPTER

Explore	11 min	the first page, the history of a misread tower level (Section 2.2), the fact on the flowmeter (Section 2.3), the try-it segments of cases 2 and 4 and the fact on derivative action (Section 2.6) and the key points
Study	70 min	the whole chapter with its three pause points and the twelve exercises at its end
Practice	30 min	the first page, the results (Section 2.6), Technology in Practice (Section 2.7), Limitations (Section 2.8) and the open challenge

The control question A tank that drains through its own bottom valve is a first-order plant whose gain and time constant both follow the square root of the level, and closing the valve makes it an integrator. What does a step test at one level predict about the loop at another, and what does one proportional loop do on each side?

Learning objectives

- Derive the resting level, the process gain, the time constant and the disturbance gain of the tank from its mass balance.
- Explain why every inlet step starts at the same slope, and measure the time constant by draining the tank.
- Predict the offset of a proportional loop from the bias it carries, and explain why it vanishes when the outlet is shut.
- Compute the valve noise that the velocity-form PID passes from the transmitter, with and without derivative action.
- Explain how a predictive controller with a model five times too sensitive still ends without offset.

Prerequisites Sections 1.4.2 and 1.4.3 (balance and linear model of a drained tank), 1.4.4 (first-order and integrating responses), 1.4.8 (step tests) and 1.6.2 (velocity form). Section 1.6.3 is needed only for the predictive controller and its estimators.

The app https://apmonitor.com/apps/level/

See also Volume 2, Chapter 2 for a gain and a time constant that fall together with speed, Volume 2, Chapter 5 for tuning an integrating level loop and Volume 2, Chapter 6 for a gain that moves too far for one tuning. Volume 7, Chapter 3 for a plant that is stable, integrating or unstable according to its operating point.

2.1 Introduction

Liquid level is the most heavily instrumented variable in a process plant and the one most often detuned on purpose. Nearly every vessel carries a level measurement because inventory has to be accounted for, pumps have to be protected against running dry, and tanks have to be protected against spilling, and a large share of those loops are detuned deliberately so that the vessel absorbs a flow variation instead of passing it downstream [1–3]. A first course introduces the subject with the pumped-outflow tank, in which discharge is set by a positive-displacement pump and is independent of the liquid above it. That vessel is a pure integrator. It has no steady state at any inflow other than the exact outflow, it does not return to a level on its own, and proportional feedback holds it at an offset fixed by the difference between the controller bias and the valve position the plant requires. A tank that drains through a valve in its own bottom behaves in the opposite way, because the head driving the discharge is the level being controlled. The vessel reaches its own resting level, and the gain and the time constant of that approach both depend on which level it is.

The two cases appear in the same chapter of every textbook and are almost never placed in the same apparatus [4, 5]. That separation obscures the behavior that governs commissioning. A gravity-drained tank is not one plant with one model. It is a family of first-order plants indexed by the operating level and by the outlet valve position, and the pumped-outflow integrator is the limiting member of that family as the outlet closes. However, the laboratory rigs and simulations used to teach level control usually fix the discharge path, so the transition between the two behaviors is described rather than measured [6]. This chapter works with a gravity-drained tank simulation in which one outlet valve moves the same vessel continuously between the self-regulating and the integrating case. It quantifies by how much a controller commissioned at one level differs from the controller the plant requires at another.

The fundamentals the experiments rest on are the ones a first course establishes, and Chapter 1 develops them on a heated tank that drains through an orifice (Section 1.4.1). A mass balance on a constant-area vessel makes the level the integral of the flow imbalance, which is conservation of mass in its simplest useful form. Bernoulli's principle applied between the free surface and the discharge makes the driving force for outflow the head itself, which is what converts an integrator

into a first-order self-regulating process [4, 5]. The experiments therefore cover integrating dynamics, process order, operating-point-dependent gain, proportional offset and its removal by integral action, the role of the controller bias, anti-windup at an actuator limit, sampled and noisy measurement, derivative action against measurement noise, and open-loop step identification of a model that is only locally valid [7]. The industrial consequence is direct. A level controller tuned at one inventory is the loop that overfills a tank or passes a disturbance downstream at another, and both outcomes have safety and economic weight [3, 8].

Section 2.2 traces level control from the float regulator to averaging level control. Section 2.3 states every relation and constant needed to rebuild the tank from this text alone: the outflow law, the two hard limits, the instrument model and the numerics, specified closely enough that a second implementation reproduces the same trajectories. Section 2.4 gives a closed-form account of the effect of the square-root outflow on a level loop. The identified gain and the identified time constant scale identically, so their ratio is a constant of the vessel, and step tests in the running application confirm it. Section 2.5 describes the velocity-form PID as implemented, with a noise propagation expression checked against the application, and three options beside it: a linear model predictive controller, a bounded moving horizon estimator for that controller and a relay experiment that writes proportional-integral constants into the PID. Section 2.6 reports the experiments. The offset of a proportional loop, the removal of that offset under integral action, and the collapse of the plant into an integrator on which proportional control has no offset at all are each produced by changing one input. The comparison of the predictive controller with the shipped proportional loop and with a proportional-integral reference separates what a plant model contributes from what integral action contributes, and the estimator and the relay are judged against the shipped choices on the same runs. Section 2.7 describes the devices and the platform of a plant level loop, and Sections 2.8 to 2.10 state the limitations, discuss the safety weight of level loops and close the chapter.

2.2 Background

The oldest surviving continuous feedback device regulated a level, and it did so in order to exploit the outflow law this application implements. The water clock attributed to Ktesibios used a float valve to hold a constant head in a supply vessel, because an orifice discharging under a constant head passes a constant flow, and only a constant flow makes the rising float in the receiving vessel a linear measure of time. Mayr traces that device and the long gap before the idea reappeared in European practice [9]. The float regulator returned as the boiler feedwater regulator and as the ball valve, and it is still the mechanism in a steam trap and in a domestic cistern. What changed in the twentieth century was the separation of the measurement from the actuator, so that a displacer or a differential-pressure transmitter reports the level to a controller that positions a valve somewhere else. The history box below tells how one such displacer misled the operators of a refinery tower.

> ⧖ HISTORY **A Level That Fell While the Tower Filled, 2005**
>
> On 23 March 2005 a refinery restarted a splitter tower 52 m tall after maintenance. For over three hours liquid was pumped in and none was taken out, and the tower overfilled. Its level transmitter was a displacer, which weighs the buoyant force on a body hanging in the liquid, and it read only a span of about 1.5 m within the bottom 2.7 m of the tower. It had been calibrated for a denser liquid than the hot liquid in the tower, so as the bottoms heated the reading fell. When the liquid stood at 48 m, the operators' screen showed 78 percent of span. Liquid released through the relief system formed a vapor cloud, and its explosion killed 15 people and injured 180 [10].

The control literature treated level as its own class of problem early, because the answers that work on temperature do not transfer. Proportional and proportional-integral level controllers were compared against nonlinear and nonconventional alternatives, and the conclusion drawn was that the right objective is often not to hold the level at all [11]. That observation became averaging level

Table 2.1 Classification of Variables. Nominal Values Are Those of the 10 m Operating Point With the Outlet Valve Half Open.

Symbol	Role	Nominal	Range
h	state, true level	10.00 m	0 to h_{max}
$\hat{h}$ (LT)	CV, measured level	10.00 m	0 to h_{max}
h_{SP}	setpoint	10.0 m	0 to h_{max}, step 0.1 m
u_{in}	MV, inlet valve	11.07 %	0 to 100 %, step 0.1 %
u_{out}	DV, outlet valve	50 %	0 to 100 %, step 1 %
q_{in}	output, inflow	0.553 m^3/s	0 to 5 m^3/s
q_{out}	output, outflow	0.553 m^3/s	≥ 0
σ_n	DV, transmitter noise 1σ	0.05 m	0 to 0.30 m
overflow	output, spill indicator	false	—

control. A surge vessel exists to absorb a flow disturbance, so a controller that removes the level deviation quickly has passed the disturbance to the next unit and defeated the purpose of the vessel. The trade between level excursion and outlet flow variation has since been posed as an optimization problem [1], solved with a predictive controller under explicit constraints on the tank limits [2], and derived directly from the mass balance so that the tuning parameter is the allowed level swing rather than a gain [12]. Luyben returned to the question forty years after the first of those papers and argued that the simple cases are simpler and the awkward cases more awkward than the literature suggests [3]. Plantwide, the requirement is weaker than regulation and stronger than nothing, because each inventory must be controlled by a flow somewhere in a consistent direction for the plant to be able to change throughput at all [13]. Tuning rules reflect the same split. An integrating process has no steady-state gain to identify, so the rules that apply to it are written in terms of a slope and a desired closed-loop time constant rather than a gain and a lag, and a self-regulating level loop and a pumped-outflow level loop are therefore commissioned by different arithmetic on the same vessel [14, 15]. Volume 2, Chapter 5 identifies and tunes such an integrating level loop on a cooling tower basin.

Tanks became a standard teaching apparatus for the same reason they are awkward. The quadruple-tank rig places two square-root outflows in a configuration whose multivariable zero can be moved from the left half plane to the right by a manual split of two flows, which is a level experiment used to teach something that has nothing to do with level [6]. The hydraulics themselves are equally standard. Draining a vessel through an orifice is the textbook application of Bernoulli's principle, and the discharge coefficient that turns the ideal Torricelli velocity into a real flow has been measured in the undergraduate laboratory for decades [16].

2.3 Process Model

The simulated vessel is an open, vertical, constant-area tank filled through a control valve at the top and drained through a valve in the bottom. The controlled variable is the level reported by transmitter LT and the manipulated variable is the inlet valve position. The outlet valve is set by the operator in both control modes, so it serves as the load disturbance and as the switch between the two dynamic characters of the vessel. There is one state, one actuator and no cascade, so every mechanism described below acts inside a single feedback path. Figure 2.1 places each symbol where the equations use it.

The signals are sorted into their roles in Table 2.1. Two hard limits bound the state. The level is clipped at the overflow height h_{max}, where an overflow indicator is raised and the excess leaves the vessel, and it is clipped at zero, where the tank is dry and the outflow term vanishes on its own. Neither limit stops the simulation. The plant is written in SI units with the level in meters, both valve positions in percent of travel and time in seconds. One state is integrated.

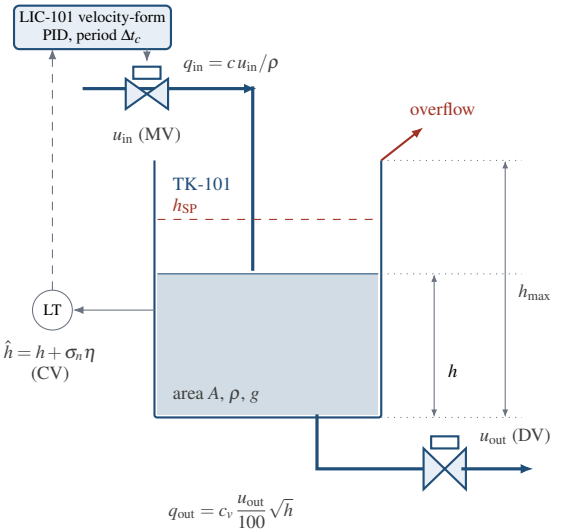


Figure 2.1 Gravity-drained tank TK-101 with its inlet control valve, its bottom outlet valve and the level transmitter. Inflow is proportional to the inlet valve position through the coefficient c, while outflow follows the square root of the level h through the outlet coefficient c_v and the fractional opening $u_{out}/100$. The controller LIC-101 reads the noisy level $\hat{h}$ and positions the inlet valve. The outlet valve is set by the operator and is never controlled.

2.3.1 Mass Balance and Outflow Law

The tank is a single well-mixed inventory whose volume is the level times a constant cross-sectional area, filled at a rate proportional to the inlet valve position and drained under its own head, as shown in Equation (2.1). It is the balance of Equation (1.2) with the two valve positions written out.

$$A\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{c\,u_{in}}{\rho} - c_v\frac{u_{out}}{100}\sqrt{h}, \qquad 0 \le h \le h_{max} \tag{2.1}$$

where A is the tank cross-sectional area in m^2, c is the inlet valve coefficient in kg/s per percent of opening, ρ is the liquid density in $\mathrm{kg/m}^3$, c_v is the outlet coefficient in m^3/s per $\mathrm{m}^{1/2}$ at full opening, and u_{in} and u_{out} are the two valve positions in percent. Both sides of Equation (2.1) carry units of m^3/s. The inlet term is written as a mass flow divided by density because the problem statement on which the application is based specifies the inlet valve in kg/s per percent, while the outlet term is volumetric and carries no density. With the default constants the balance reduces to $\mathrm{d}h/\mathrm{d}t = 0.05\,u_{in} - 0.35\,(u_{out}/100)\sqrt{h}$ in m/s.

The outflow term is the Torricelli form obtained from Bernoulli's principle applied between the free surface and the discharge, with the ideal velocity $\sqrt{2gh}$ multiplied by an orifice area and a discharge coefficient, as shown in Equation (2.2).

$$q_{out} = C_d A_o\sqrt{2gh} \;\equiv\; c_v\sqrt{h} \qquad \text{at } u_{out} = 100\,\% \tag{2.2}$$

where C_d is the discharge coefficient, A_o is the discharge area in m^2 and g is gravitational acceleration. Neither C_d, A_o nor g appears anywhere in the code. The three are lumped into the single adjustable constant $c_v = C_d A_o\sqrt{2g}$, which is the only form a controller can distinguish. The default $c_v = 0.35\,\mathrm{m}^3/\mathrm{s}$ per $\mathrm{m}^{1/2}$ implies $C_d A_o = 0.0790\,\mathrm{m}^2$, which is 7.9 % of the $1\,\mathrm{m}^2$ tank cross-section and is a larger drain than a tank of this footprint would carry. The choice is deliberate, because it empties a full tank in about 20 s and makes every experiment fit inside a laboratory period. Such a drain would not be fitted to a 12 m vessel of $1\,\mathrm{m}^2$, but it compresses the whole self-regulating response

into a window that can be watched in real time without a speed multiplier, and the scaling results of Section 2.4 do not depend on it. The valve opening enters as a linear multiplier on c_v, which is a linear installed characteristic on the effective discharge area. Downstream pressure, pipe friction and the elevation of the outlet nozzle above the tank floor are absent, so the head that drives the discharge is the level itself.

> **+ DID YOU KNOW? The Square Root Inside a Flowmeter**
>
> The square root that drains this tank also governs a common flowmeter. Flow through an orifice plate is proportional to the square root of the pressure drop divided by the density, so the square root of the differential pressure signal is taken before it reads as flow [17]. Two such drains in series form the plant of Chapter 3.

Setting the derivative in (2.1) to zero gives the resting level at any pair of valve positions, as shown in Equation (2.3).

$$\bar{h} = \left[\frac{100\, c\, \bar{u}_{\text{in}}}{\rho\, c_v\, \bar{u}_{\text{out}}} \right]^2 \tag{2.3}$$

Equation (2.3) is the whole difference between this vessel and a pumped-outflow tank. A resting level exists for every inlet setting as long as the outlet valve is open at all, and it grows as the square of the inlet position, so the operating map is steep. Holding 10 m against a half open outlet requires $\bar{u}_{\text{in}} = 11.07\,\%$, and holding the same level against a fully open outlet requires 22.14 %. Beyond $\bar{u}_{\text{in}} = 24.25\,\%$ with the outlet wide open the resting level lies above $h_{\max}$ and the tank overflows instead of settling.

2.3.2 Measurement

The controller never reads the state. It reads the transmitter output of Equation (2.4).

$$\hat{h} = h + \sigma_n\, \eta \tag{2.4}$$

where σ_n is the noise standard deviation in m and η is a standard normal deviate drawn once per control execution by the Box-Muller transform from a seeded mulberry32 generator. The noise is white at the 10 Hz control rate, with no transmitter lag, no bias and no drift. The default of 0.05 m is pessimistic for a modern level instrument and is set that way so that the cost of derivative action is visible in a single run. The instrument chain contains no nonlinearity at all, and in particular the reported value is not clipped at zero, so an empty tank reads negative half the time. Half-rectifying the reading instead would add a bias of $\sigma_n/\sqrt{2\pi} = 0.399\,\sigma_n$ near an empty tank, which is 0.12 m at the largest noise setting and is a systematic error introduced by the instrument model rather than by the instrument. The true level and the measured level are reported together, so the effect of the measurement on the loop is separable from the state that produced it.

Every constant needed to rebuild the simulator is listed in Table 2.2. The initial level is the steady state of the settings a case starts with, and the initial integral is the manual inlet position at load, as described in Section 2.5. Operator-entered constants are sanitized before use, with A floored at machine epsilon, c and c_v floored at zero, $h_{\max}$ floored at 1 m and σ_n floored at zero, and the level is re-clipped to a new tank height at the next control execution. The setpoint range follows the tank height, and lowering $h_{\max}$ below the setpoint pulls the setpoint down to it.

2.3.3 Numerics

Equation (2.1) is integrated with explicit Euler on a fixed 0.05 s step, one of the app-supplied steps that Section 1.6.1 allows in place of the framework's Runge–Kutta method. A first-order right-hand side with a square root and two clipping bounds does not reward a higher-order integrator, because the truncation error is dominated by the two limits rather than by the smooth part. The level is clipped to $[0, h_{\max}]$ after every step, and the overflow indicator records whether the unclipped value exceeded

Table 2.2 Model and Controller Parameters. Every Value in the First Two Blocks Is Adjustable in the Application.

Symbol	Meaning	Default	Range	Unit
Plant constants, operator adjustable				
A	tank cross-section	1.0	0.3–5.0	m^2
c	inlet valve coefficient	50	10–100	kg/(s %)
c_v	outlet coefficient	0.35	0.10–0.80	m^3/s per $m^{1/2}$
h_{max}	overflow height	12	5–20	m
σ_n	transmitter noise	0.05	0–0.30	m
Controller, setpoint and outlet, operator adjustable				
K_c	proportional gain	10	0–100	%/m
K_I	integral gain	0	0–20	%/(m s)
K_D	derivative gain	0	0–50	% s/m
h_{SP}	level setpoint	10	0–h_{max}	m
u_{out}	outlet valve	10	0–100	%
—	speed multiplier	1	0.5–10	—
—	MPC estimator	Kalman	or MHE	—
Fixed in the source				
N_e	MHE window	30	—	executions
ε_r	relay band	0.15	—	m
ρ	liquid density	1000	—	kg/m^3
Δt	integration step (Euler)	0.05	—	s
Δt_c	controller sample period	0.1	—	s
I_0	initial integral	u_m	—	%
T_t	anti-windup tracking time	K_c/K_I	—	s
—	historian rate	5	—	samples/s
$h(0)$	initial level	steady	—	m

h_{max} on that step. The controller executes every 0.1 s of simulated time, which is two integration steps, and one row is recorded every 0.2 s, at every second control execution. A row carries the measured level that the controller acts on at the stamped time and the valve position held over the control period that has just ended. All three clocks run on simulated time, and the noise generator is reseeded whenever a case is loaded or reset, so a run at any speed between 0.5 and 10 gives the same trajectory with noise enabled (Section 1.6.1).

2.3.4 Assumptions

The model keeps only the mechanisms that change how the loop must be tuned. Each assumption below is stated with the range over which it holds and the condition under which it fails.

1. *Quasi-steady outflow.* The discharge is the steady Torricelli expression evaluated at the instantaneous level, so the liquid in the outlet line has no inertia and responds to a level change without delay. This is accurate while the drain time is long compared with the acceleration time of the outlet column, which holds throughout the default parameter set. It fails for a long discharge line, for a rapid valve stroke, and for any question about water hammer or oscillation of the outlet column.

2. *Constant discharge coefficient.* C_d is folded into c_v and does not vary with level or with valve opening. Measured discharge coefficients move by several percent over a wide Reynolds range and with the geometry of the opening [16]. The consequence is that the simulated outflow follows $\sqrt{h}$ more exactly than a real drain does, which makes the scaling results of Section 2.4 cleaner than a physical rig would show them.

3. *Linear valve characteristics.* The inlet flow is strictly proportional to the inlet position and the outlet coefficient is strictly proportional to the outlet position. There is no installed characteristic, no rangeability limit, no seat leakage, no stroking time, no hysteresis and no stiction. A real level loop frequently oscillates because of the last of these rather than because of its tuning, and no tuning in this application produces that symptom.

4. *Constant area and a flat bottom.* The cross-section A does not vary with height, so the level and the inventory are proportional. A conical, spherical or horizontal cylindrical vessel adds a second, purely geometric nonlinearity on top of the square-root outflow, and none of them is represented.

5. *Overflow discards liquid.* Above $h_{\max}$ the state is clipped and the excess leaves the model. Mass is therefore not conserved during a spill, the spilled rate is not reported, and the level returns from the limit as soon as the balance turns negative. This is adequate to show that a loop has lost control of the inventory. It cannot be used for any question about containment volume or spill rate.

6. *Adequate integration step.* The fastest plant reachable from Table 2.2 is $A = 0.3\,\mathrm{m}^2$ with $c_v = 0.8$ and the outlet fully open, which gives $\tau = 0.75\sqrt{\bar{h}}$ in seconds, so the 0.05 s step is at most 6.7 % of the local time constant above 1 m of level. Accuracy degrades close to an empty tank, where $\partial\sqrt{h}/\partial h$ is unbounded, and the clip at zero absorbs the resulting undershoot.

7. *Ideal transmitter apart from noise.* LT has no lag, no bias, no drift, no resolution limit and no rangeability, and its noise is white and Gaussian. Real level transmitters are dominated by low-frequency effects such as density change and turbulence at the surface, which are correlated rather than white, and a differential-pressure transmitter reads a head that depends on the density it was calibrated for.

8. *Constant density.* ρ is fixed at $1000\,\mathrm{kg/m}^3$ in the source and is not exposed. A change in composition or temperature that alters the density would move the inlet mass flow, the head at the outlet and the transmitter calibration together, and none of the three is available.

9. *Omitted entirely.* Inlet flow disturbances upstream of the valve, downstream pressure on the discharge, vapor space pressure, agitation, heat effects, cascade to a flow controller and any second vessel are all absent. Each is omitted because none of them changes the single behavior the application exists to expose.

2.4 Dynamic Analysis

2.4.1 Linearization About an Operating Level

Expanding Equation (2.1) about a resting point $(\bar{h}, \bar{u}_{\mathrm{in}}, \bar{u}_{\mathrm{out}})$ and keeping first-order terms, as Section 1.4.3 does for the running example, gives the first-order deviation model of Equation (2.5).

$$\tau \frac{\mathrm{d}h'}{\mathrm{d}t} + h' = K_u\, u'_{\mathrm{in}} + K_d\, u'_{\mathrm{out}} \tag{2.5}$$

where the primes denote deviations from the resting point and the three constants are given by Equation (2.6).

$$\tau = \frac{2A\sqrt{\bar{h}}}{c_v\, \bar{u}_{\mathrm{out}}/100}, \quad K_u = \frac{c}{\rho A}\,\tau, \quad K_d = -\frac{2\bar{h}}{\bar{u}_{\mathrm{out}}} \tag{2.6}$$

where τ is the process time constant in s, K_u is the process gain in m/% of inlet travel and K_d is the disturbance gain in m/% of outlet travel. Both τ and K_u carry a factor $\sqrt{\bar{h}}$, so they grow together as the tank fills, and both carry $1/\bar{u}_{\mathrm{out}}$, so they grow together as the drain is throttled. At 10 m with the

outlet half open the tank is a first-order plant with $K_u = 1.807\,\mathrm{m}/\%$ and $\tau = 36.1\,\mathrm{s}$, and at 3 m with the same outlet setting the same tank is a first-order plant with $K_u = 0.990\,\mathrm{m}/\%$ and $\tau = 19.8\,\mathrm{s}$. The two descriptions differ only in the inventory the vessel holds. The running example of Section 1.4.3 shows the same scaling in Equation (1.5).

Two consequences follow directly. The first is that the loop gain of a fixed controller is proportional to $\sqrt{\bar{h}}$, so a proportional gain set at the bottom of the tank produces a loop gain $\sqrt{h_{\max}/h_{\min}}$ times larger at the top. Over the usable range of the default tank, from 1 m to 12 m, that factor is 3.46. The neutralizer of Volume 2, Chapter 6, whose gain moves by orders of magnitude over one valve stroke, is the case in which a gain schedule becomes necessary. The second is that τ is exactly the time the same tank takes to empty from $\bar{h}$ with the inlet shut, because integrating (2.1) with $u_{\mathrm{in}} = 0$ gives $t_{\mathrm{empty}} = 2A\sqrt{h_0}/(c_v \bar{u}_{\mathrm{out}}/100)$. The identified time constant of the level loop and the observed drain time of the vessel are the same number, so the model is verified by a stopwatch measurement.

2.4.2 Invariance of the Gain to Time Constant Ratio

Dividing the gain by the time constant in (2.6) removes every operating-point term, as shown in Equation (2.7).

$$\frac{K_u}{\tau} = \frac{c}{\rho A} = 0.05\,\mathrm{m}/(\mathrm{s}\,\%) \tag{2.7}$$

The right-hand side of Equation (2.7) is the initial slope of the level response to an inlet step, and it depends on neither the level nor the outlet position. Every step test on this tank therefore starts at the same slope and differs only in its final value and in the time taken to reach it. That slope is also the integrating gain of the vessel, which is the connection to the second configuration. The cruise control of Volume 2, Chapter 2 has the same property: drag moves its gain and its time constant together, and their ratio is the thrust-to-mass ratio of the car.

🕮 **Pause.** Try Exercise 2.1 before reading on.

2.4.3 The Integrating Limit

Closing the outlet valve removes the only level-dependent term in (2.1) and leaves Equation (2.8).

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{c\,u_{\mathrm{in}}}{\rho A} = 0.05\,\mathrm{m}/(\mathrm{s}\,\%) \times u_{\mathrm{in}} \tag{2.8}$$

The vessel is now a pure integrator with no resting level at any inlet setting except zero, which is the pumped-outflow tank of the textbook with the pump stopped. The application offers no separate integrating configuration and requires none, because $u_{\mathrm{out}} = 0\,\%$ produces it exactly and the outlet valve position moves continuously between the two cases. Taking $\bar{u}_{\mathrm{out}} \to 0$ in (2.6) sends τ and K_u to infinity with their ratio fixed at (2.7), which is the standard limiting argument by which a first-order lag with a very long time constant is treated as an integrator [7] and Equation (1.9) is reached from the running example of Chapter 1. The spent-fuel pool of Chapter 9 moves between the same two characters when its cooling trains stop. The consequence for control is asymmetric. The inlet valve can add liquid and no actuator can remove it, so a controller that has driven the level above setpoint with the outlet shut cannot return the level to setpoint.

2.4.4 Identification From the Running Application

Table 2.3 reports three step tests performed in manual mode from the first shipped case with the outlet valve wide open and the transmitter noise set to zero, each a 1 % increase in the inlet valve at $t = 300\,\mathrm{s}$ from the resting level at which the case starts once the two valve positions are entered before the run. The gain is the secant slope over the step, and the time constant is a one-parameter least-squares fit of a first-order response to the 300 s of logged history that follows the step, with the initial and final levels fixed at their measured values.

Table 2.3 Step Tests at Three Operating Levels, Outlet Valve at 100 % and $\sigma_n = 0$. The Predicted Columns Come From Equation (2.6) Evaluated at the Starting Level.

u_{in} (%)	$\bar{h}$ (m)	$\hat{K}_u$ (m/%)	K_u (m/%)	$\hat{\tau}$ (s)	τ (s)	$\Delta h/\bar{h}$ (%)	$\varepsilon_{\max}$ (m)
$10 \to 11$	2.041	0.429	0.408	8.73	8.16	21.0	0.0015
$17 \to 18$	5.898	0.714	0.694	14.44	13.88	12.1	0.0015
$22 \to 23$	9.878	0.918	0.898	18.53	17.96	9.3	0.0015

The identified gain rises by a factor of 2.14 between 2.04 m and 9.88 m against the factor of 2.20 predicted by $\sqrt{9.878/2.041}$, and the identified time constant rises by a factor of 2.12 against the same prediction. Both fitted values sit above the linearized prediction because a 1 % step moves the level by 0.43 m at 2.04 m, which is 21.0 % of the starting level, so the response is measurably faster at the end than at the beginning and the fit splits the difference. The bias is largest at the lowest level for that reason, at 7.0 % of τ against 3.2 % at the highest, while the largest residual of the fitted first-order response is 0.0015 m in all three tests, so the identified pair is a good description of a response that is nonetheless not first order. A drain test from a full 12 m tank with the inlet shut and the outlet wide open empties the vessel in 19.65 s against the 19.79 s of the analytic expression, and passes three quarters, one half and one quarter of the initial level at 2.65, 5.80 and 9.90 s against 2.65, 5.80 and 9.90 s, all read at the 0.05 s integration step. The 0.14 s shortfall arises close to the empty tank, where $\partial\sqrt{h}/\partial h$ is unbounded and the explicit Euler step is least accurate.

Three distinct models appear in this chapter, the three that Section 1.4.9 separates. The simulation model is Equation (2.1), nonlinear and exact within its assumptions. The identified control model is the first-order pair $(\hat{K}_u, \hat{\tau})$ of Table 2.3, valid near one level. The PID holds no model at all, because the algorithm of Section 2.5 carries three gains and a sample period and no representation of the plant. The predictive controller of Section 2.5.5 holds the third model, a first-order pair identified by a step test at 10 m with the outlet 10 % open and never updated as the plant moves away from that point.

2.5 Controller Design

2.5.1 Velocity Form With Derivative on the Measurement

LIC-101 runs the velocity-form PID of Section 1.6.2 in gain form [15]. Each cycle it computes an increment to the inlet valve rather than the position itself, and the operating point is carried by an integral state rather than by an operator-entered bias. The source stores the manual valve position at case load as a fixed bias and starts its integral at zero, and the equations below fold that constant into the integral, which gives the same recursion. The error is the setpoint minus the measurement, as shown in Equation (2.9), and the three increments are given by Equations (2.10) to (2.12).

$$e_k = h_{\text{SP}} - \hat{h}_k \tag{2.9}$$

$$\Delta u_k^P = K_c\, e_k - K_c\, e_{k-1} \tag{2.10}$$

$$\Delta u_k^I = K_I\, e_k\, \Delta t_c \tag{2.11}$$

$$\Delta u_k^D = -K_D\, \frac{\hat{h}_k - 2\hat{h}_{k-1} + \hat{h}_{k-2}}{\Delta t_c} \tag{2.12}$$

where K_c is in %/m, K_I in %/(m s), K_D in % s/m and $\Delta t_c = 0.1$ s. The increments are differences of the complete proportional and derivative terms, so a gain edited during a run acts on the whole current error at once. The proportional and derivative increments are accumulated, the demand is clipped to

the valve travel, and the integral is updated afterwards, as shown in Equation (2.13).

$$\begin{aligned} s_k &= s_{k-1} + \Delta u_k^P + \Delta u_k^D \\ c_k &= s_k + I_{k-1} \\ u_k &= \min(100,\ \max(0,\ c_k)) \\ I_k &= I_{k-1} + \Delta u_k^I + \frac{\Delta t_c}{T_t}\,(u_k - c_k) \end{aligned} \tag{2.13}$$

Here s_k accumulates the proportional and derivative increments without limit, c_k is the output the algorithm demands (the sum S_k and the unlimited output v_k of Equations (1.25) and (1.26), with the bias folded into I), the integral used at execution k is the one accumulated through execution $k-1$, and the last term is a back-calculation correction with tracking time $T_t = K_c/K_I$, which is 10 s at the tuning used in Section 2.6. The integral is updated only when $K_I \neq 0$. Away from both limits $u_k = c_k$, the correction vanishes, and the recursion is algebraically identical to the positional law $u_k = I_0 + K_c e_k + K_I \sum_{j<k} e_j \Delta t_c$ plus the derivative term, with I_0 the initial integral, so every tuning quoted in this chapter has its positional meaning. At a limit the two differ, because only the integral is corrected. Keeping s_k unclipped matters most at $K_I = 0$, where there is no integral to absorb the clipped part, for the reason Section 1.6.2 gives, and Section 2.6.3 shows what clipping it would do on this tank. The derivative acts on the measurement with no filter, so a setpoint move reaches the valve only through the proportional term, as a step of $K_c \Delta h_{\mathrm{SP}}$. The loop is reverse acting, because opening the inlet raises the level and e_k falls, and no action switch is offered. The gains are entered as K_c, K_I and K_D rather than as K_c, τ_I and τ_D, so a tuning correlation expressed in time constants has to be converted through $K_I = K_c/\tau_I$ and $K_D = K_c\tau_D$ first.

2.5.2 Initialization, Transfer and the Effective Bias

Every case starts at steady operation. In automatic the tank is placed at the setpoint and the manual station at the inlet position that holds it there, $\bar{u}_{\mathrm{in}}$ from Equation (2.3) with the outlet setting and plant constants in force, and in manual the tank is placed at the resting level of the manual inlet position. With the outlet shut no open inlet has a resting level, and the case starts at the setpoint, with the inlet shut in automatic. A setting changed before the run re-initializes the case exactly as Reset would, so a setpoint, outlet setting or inlet position entered before Run moves the starting steady state rather than producing a step. When a case is loaded the integral is set to the manual inlet position, so a case that opens in automatic issues $\bar{u}_{\mathrm{in}}$ plus K_c times its first error, which is transmitter noise alone. In manual the controller keeps executing and tracks the operator, setting $I_k = u_m - s_k$ so that its demand equals the manual valve position u_m. In automatic the manual setting follows the controller output at every execution without rounding, so both transfers are bumpless, as Section 1.6.2 describes.

The tracking gives the transfer a consequence that a positional law with an operator-entered bias does not have. Summing (2.10) from the last manual execution with $K_I = K_D = 0$ leaves $u_k = u_0 + K_c(e_k - e_0)$, where u_0 and e_0 are the valve position and the error at that execution. The loop therefore behaves as a positional law whose bias $u_0 - K_c e_0$ is fixed by the transfer condition. A shut-outlet tank holds still only with the valve shut, so a proportional-only loop transferred there rests at $\hat{h}_0 + u_0/K_c$, where $\hat{h}_0$ is the level read at transfer, whatever the setpoint. Measured on the shut-outlet tank at $K_c = 10\,\%/\mathrm{m}$ with a 10 m setpoint and no transmitter noise, starting from an empty tank (the first case with the inlet shut and the outlet wide open before the run, the outlet closed and the inlet opened at the execution at $t = 0.1$ s), filling by hand with the inlet at 0, 4, 20 and 60 % and switching to automatic as the level read 5.20, 5.16, 5.10 and 5.10 m settled the loop at 5.20, 5.56, 7.10 and 11.10 m, which matches $\hat{h}_0 + u_0/K_c$ to better than 0.001 mm in all four cases. For $u_0 = 0$ the tank was filled at 20 % and the inlet shut 1 s before the transfer. A case that opens in automatic carries the bias of its own starting inlet position, 2.21 % in the cases that start at 10 m with the outlet 10 % open.

2.5.3 Back-Calculation as the Anti-Windup Mechanism

Anti-windup is the last term of Equation (2.13) rather than a separate stage, and the running sum s_k is never truncated. With $K_D = 0$ the sum s_k telescopes to $K_c e_k$, so on a limit $u_{\lim}$ the clipped part is $u_{\lim} - c_k = u_{\lim} - K_c e_k - I_{k-1}$, and with $T_t = K_c/K_I$ the error part of the correction cancels Δu_k^I exactly. What remains is Equation (2.14).

$$I_k - u_{\lim} = \left(1 - \frac{\Delta t_c}{T_t}\right)(I_{k-1} - u_{\lim}) \tag{2.14}$$

where $u_{\lim}$ is 0 % or 100 %. The integral relaxes to the limit itself with time constant T_t whatever the error, and the demand exceeds the limit only by the proportional term $K_c e_k$. Once the integral has relaxed, the valve leaves the limit when the error changes sign, and no stored integral has to be worked off. Case 8 run with $K_I = 1\,\%/(\mathrm{m\,s})$ holds the valve at 100 % from the setpoint move at $t = 5\,\mathrm{s}$ to the end of a 60 s run (inlet coefficient 10 kg/s per percent, outlet wide open, setpoint moved from 6 to 10 m), and it reproduces the factor $(1 - \Delta t_c/T_t)^{100} = 0.366$ over 10 s to five digits. The application offers no free integrator, so it demonstrates the absence of windup rather than the failure mode.

2.5.4 Noise Propagation Through the Velocity Form

The derivative increments telescope in the same way. Summing (2.12) leaves a first difference rather than a running sum, so the noise contribution to the output is bounded rather than a random walk, as shown in Equation (2.15).

$$\sigma_u = \sigma_n \sqrt{\left(K_c + \frac{K_D}{\Delta t_c}\right)^2 + \left(\frac{K_D}{\Delta t_c}\right)^2} \tag{2.15}$$

where σ_u is the standard deviation of the valve position about its resting value in percent, valid while the loop is quiet and the output is off both limits. Equation (2.15) is obtained by writing the accumulated noise contribution as $-(K_c + K_D/\Delta t_c)\, n_k + (K_D/\Delta t_c)\, n_{k-1}$ with n_k the independent noise sample at step k. The short sample period amplifies the derivative term. With the default $\Delta t_c = 0.1\,\mathrm{s}$, a derivative gain of $K_D = 1\,\%\,\mathrm{s/m}$ contributes a factor $K_D/\Delta t_c = 10\,\%/\mathrm{m}$, which equals the whole default proportional gain. Integral action contributes a slow random walk of standard deviation $K_I \sigma_n \Delta t_c \sqrt{k}$ after k samples, which at $K_I = 1\,\%/(\mathrm{m\,s})$ reaches 0.17 % after two simulated minutes and is negligible beside the other two terms.

2.5.5 Model Predictive Control

A model predictive controller (MPC) is offered as an alternative to the PID in automatic mode, and the PID remains the default of every case. Every $T_s = 0.5\,\mathrm{s}$, or five controller executions, the MPC plans the inlet valve over a horizon, applies the first move and holds it until the next decision, the receding-horizon scheme of Section 1.6.3 [18, 19]. Its model is Equation (2.16), fitted with the noise off to the averaged responses to inlet steps of ± 0.2 percentage points at the design point of the shipped cases, $\bar{h}_m = 10\,\mathrm{m}$ with the outlet 10 % open and $\bar{u}_m = 2.214\,\%$. The fitted dead time is zero and the root-mean-square residual is 0.0004 m.

$$\tau_m \frac{\mathrm{d}x}{\mathrm{d}t} = -x + K_m\,(u_{\mathrm{in}} - \bar{u}_m + d)\,, \qquad h_m = \bar{h}_m + x \tag{2.16}$$

where x is the level deviation of the model in m, $K_m = 9.030\,\mathrm{m}/\%$ and $\tau_m = 180.7\,\mathrm{s}$ are the fitted gain and time constant, d is an unmeasured disturbance in percent of inlet travel and h_m is the level the model predicts. Equation (2.6) gives 9.035 m/% and 180.7 s at that point, and $K_m/\tau_m = 0.0500\,\mathrm{m}/(\mathrm{s}\,\%)$ reproduces the invariant of Equation (2.7). The model contains neither the square-root outflow nor the outlet valve. With the outlet half open the same tank at 10 m has $K_u = 1.807\,\mathrm{m}/\%$ and $\tau = 36.1\,\mathrm{s}$, a factor of five below both.

The disturbance d is estimated by a steady-state Kalman filter on Equation (2.16), sampled for a zero-order hold at Δt_c and augmented with d as a random walk, which is the standard arrangement for offset-free predictive control [20]. At every execution, whichever controller is in charge, the filter predicts and corrects as shown in Equation (2.17).

$$\begin{aligned} \bar{x}_k &= a\,\hat{x}_{k-1} + b\left(u_{k-1} - \bar{u}_m + \hat{d}_{k-1}\right) \\ \hat{x}_k &= \bar{x}_k + \ell_x\left(\hat{h}_k - \bar{h}_m - \bar{x}_k\right) \\ \hat{d}_k &= \hat{d}_{k-1} + \ell_d\left(\hat{h}_k - \bar{h}_m - \bar{x}_k\right) \end{aligned} \tag{2.17}$$

where $a = e^{-\Delta t_c/\tau_m} = 0.999447$, $b = K_m(1-a) = 0.004995\,\mathrm{m/\%}$, u_{k-1} is the inlet position applied over the period just ended, $\hat{h}_k$ is the reading the PID also uses, and $\ell_x = 0.0569$ and $\ell_d = 0.340\,\%/\mathrm{m}$ are the steady-state gains for a measurement variance $r = \sigma_n^2 = 0.0025\,\mathrm{m}^2$, a state noise variance $10^{-6}r$ and a disturbance noise variance $q_d = 10\,r/K_m^2 = 3.07\times10^{-4}\,\%^2$. A load that the model lacks appears in $\hat{d}$ as the inlet offset that would produce it, and the plan holds $\hat{d}$ over its horizon, so the closed loop settles without offset. The estimate is integral action obtained through the model rather than through the error.

The plan is a quadratic program on the model sampled at T_s, over a horizon $T_p = 360\,\mathrm{s}$ of $N = 720$ steps, or two model time constants. It chooses the inlet positions $v_1,\ldots,v_{12}$, held over move blocks of $1,1,1,1,2,2,3,4,6,9,15$ and 675 steps, that minimize the objective of Equation (2.18) subject to $0 \le v_j \le 100$.

$$J = \frac{T_s}{\tau_n}\sum_{k=1}^{N}\left(\frac{h_{m,k} - h_{\mathrm{SP}}}{y_s}\right)^2 + \lambda\sum_{j=1}^{12}\left(\frac{v_j - v_{j-1}}{u_s}\right)^2 \tag{2.18}$$

where $h_{m,k}$ is the level predicted k steps ahead with $\hat{d}$ held, v_0 is the inlet position applied now, $\tau_n = 180.7\,\mathrm{s}$ keeps the sum from growing with the horizon, $y_s = 0.5\,\mathrm{m}$ is the settling band of the scorecard, $u_s = 100\,\%$ is the valve span and $\lambda = 0.03$ is a dimensionless move suppression. The valve range, in percent, is the only constraint and it is hard. There is no level band, overflow limit, rate limit or terminal weight, and no preview, because the controller is not told the outlet position. The decisions are formed as deviations from $\bar{u}_m$, and a ridge of 10^{-9} on the scale of the weights keeps the program strictly convex. Its 12 variables and 24 bound rows are handled by a dual active-set solver [21] from the framework's shared modules. The shipped λ and the factor of 10 in q_d come from one rule for the seven single-loop studio apps, which takes the largest move suppression within 10 % of the best true-level IAE among settings with no more valve travel than the shipped PID on cases 3 and 4. For this tank the rule selects the smallest λ swept.

A plan that is not optimal, or whose first move is not finite or leaves the valve range, is rejected. The PID tracks the applied valve at every execution as it tracks the operator in manual, so it takes the valve without a bump, the history is marked, and the next valid plan takes the valve back. A rejection forced at 6 s of case 9 without noise hands over with a valve change of 0.38 %, the size of the next three moves of the PID (0.32 to 0.36 %), and the take-back moves the valve by 3.2 %. The first plan is solved when a case loads, so a run under the MPC is logged from $t = 0$. The controller reads only the transmitter, the applied inlet position, the setpoint and the valve limits, which keeps the plant, the instrument and the controller model apart.

2.5.6 Moving Horizon Estimation

While the MPC is the controller, its plans may start from a moving horizon estimator (MHE) instead of the filter of Equation (2.17), which keeps running either way. At execution k the MHE chooses z_s and $w_s,\ldots,w_{k-1}$ over the window $j = s,\ldots,k$, $s = k - N_e$, to solve the bounded least-squares problem

of Equation (2.19), in the form analyzed in [22].

$$\begin{aligned} \min\ & \|z_s - \bar{z}_s\|^2_{\bar{P}^{-1}} + \sum_{j=s}^{k-1} w_j^{\mathsf{T}} Q^{-1} w_j \\ & + \frac{1}{r} \sum_{j=s}^{k} \left(\hat{h}_j - \bar{h}_m - x_j\right)^2 \\ \text{s.t. } & z_{j+1} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} z_j + \begin{bmatrix} b \\ 0 \end{bmatrix} (u_j - \bar{u}_m) + w_j \\ & 0 \le \bar{h}_m + x_j \le h_{\max,0} \end{aligned} \tag{2.19}$$

Here $z_j = [x_j \;\; d_j]^{\mathsf{T}}$ augments the model state with the disturbance, w_j is the process noise, $Q = \mathrm{diag}(10^{-6} r, q_d)$ and r are the variances of the filter, u_j is the inlet position applied after execution j, $\bar{z}_s$ is the estimate of execution $s-1$ advanced one step, and $\bar{P}$, the steady a-priori covariance of the filter, is the fixed arrival cost. The window of $N_e = 30$ executions spans 3 s, which gives 62 variables and 62 bound rows for the dual active-set method of the plan. The bound holds the model level between the floor and $h_{\max,0} = 12\,\mathrm{m}$, the declared default rather than the height set for the run, and d is unbounded because no declared value limits it. Without an active bound the problem is the one the filter solves recursively. A window that ends without an optimum, or with an estimate that is not finite or not inside the bound, gives way to the Kalman correction of the MHE's own prediction one step ahead, projected onto the bound, and the history marks the substitution.

2.5.7 Relay Autotuning

The PID gains may instead be obtained from the closed loop itself through the relay feedback experiment of Åström and Hägglund [23]. It is an action started from the operating state of the moment under the PID in automatic, and no shipped case runs with it engaged. The relay holds the inlet valve δ_r above or below u_0, the mean of the valve positions applied during the 6 s before the test, with δ_r the smaller of 5 % of the span and 0.9 of the distance from u_0 to the nearer limit, and it switches each time the reading passes more than $\varepsilon_r = 3\sigma_{n,0} = 0.15\,\mathrm{m}$ beyond the setpoint, three times the declared noise $\sigma_{n,0}$ whatever noise the run carries. Cycles with a half shorter than four executions are not counted. When the periods and the swings $\pm\alpha$ of two successive counted cycles match to 5 %, the hysteresis-corrected describing function yields the ultimate gain K_{cu}, and the Tyreus–Luyben settings follow from it and the period P_u, as in Equation (2.20), where D is the part of each cycle that the valve spends above u_0, equal to one half when the cycle is symmetric.

$$K_{cu} = \frac{4\delta_r \sin(\pi D)}{\pi \sqrt{\alpha^2 - \varepsilon_r^2}}, \qquad K_c = \frac{K_{cu}}{3.2}, \qquad K_I = \frac{K_c}{2.2\,P_u} \tag{2.20}$$

The constants enter the PID to three significant figures with $K_D = 0$ and stay editable. Without convergence the relay stops after 30 cycles or 120 s, a second press cancels it, and the PID tracks the applied valve throughout so that it resumes without a bump. Once $\psi = \arcsin(\varepsilon_r/\alpha)$, the angle by which the hysteresis displaces the cycle from the -180° point, reaches 10°, the result is reported as set by the band rather than by the tank.

2.6 Results

The application ships ten cases. All ten start at steady operation, nine of them at 10 m with the outlet 10 % open and the inlet at the 2.21 % that holds that level, and the first scripted event of every case but the first falls at $t = 5\,\mathrm{s}$, so nothing moves before it except the response of the loop to transmitter noise. Case 1, Reference operating point, holds that state in manual. Case 2, Open-loop actuator step, steps the inlet from 2.21 to 1.21 % in manual. Cases 3, 6 and 7 move the setpoint in automatic, to 11.2 m at

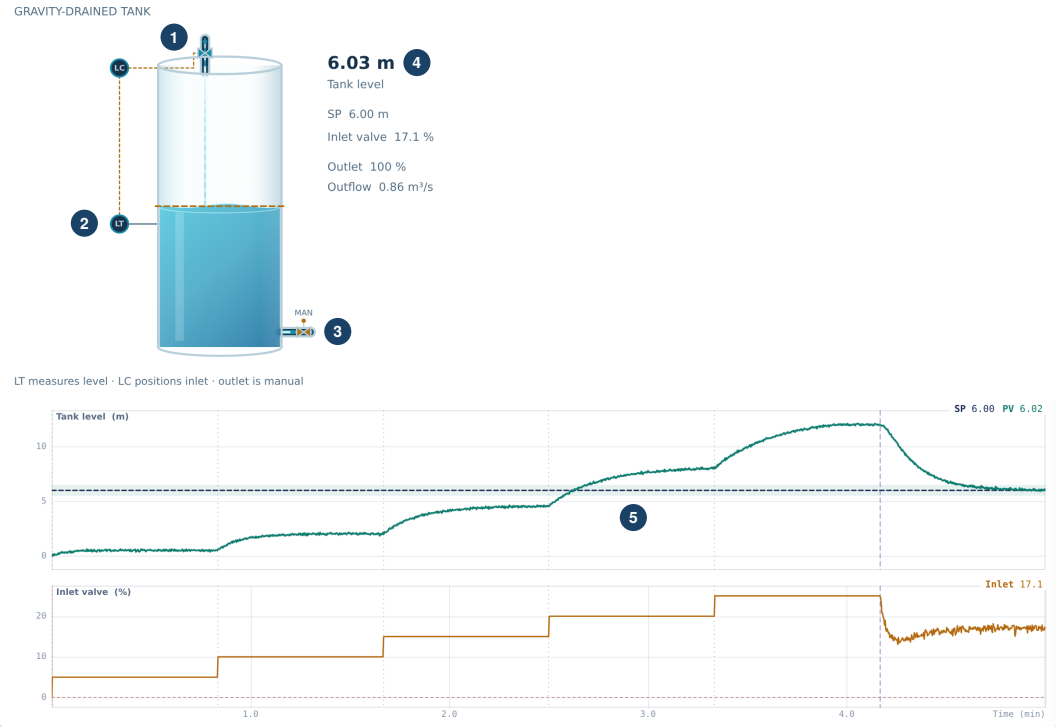


Figure 2.2 Equal inlet valve steps producing unequal level steps on the gravity-drained tank, run from the first shipped case with the inlet shut and the outlet wide open before the run, so that the case starts with the tank empty. The outlet valve stays wide open for the whole run and the transmitter noise is at its 0.05 m default. Manual inlet settings of 5, 10, 15, 20 and 25 %, the first entered at $t = 0.1$ s and each later one 50 s after the one before, end at levels of 0.51, 2.04, 4.55 and 8.01 m and then at the 12 m overflow limit, reached at $t = 235.8$ s, so the same 5 % increment raises the level by 0.51 m near the bottom of the tank and by 3.46 m near the top, and each new level takes longer to reach than the one before it. At $t = 250$ s the loop is switched to automatic with $h_{SP} = 6$ m, $K_c = 10\,\%/\text{m}$ and $K_I = 1\,\%/(\text{m s})$, and at $t = 300$ s it has drained the tank to 6.03 m with the inlet valve at 17.1 %. The figure is a composite of the application window: the plant view above the two trend strips, with the header and the side panels cut. Callout 1 marks the inlet valve that LC moves, 2 the level transmitter LT, 3 the outlet valve set by hand, 4 the readout column with the level of 6.03 m and the inlet at 17.1 %, and 5 the level trend, whose steps grow at each equal valve step. The trend draws the levels without printing them, and their values are those given in Section 2.6.

the reference gain $K_c = 10\,\%/\text{m}$ in case 3 and to 10.96 m at $K_c = 4$ and 20 %/m in the conservative and the aggressive case. Cases 4 and 5 open the outlet from 10 to 50 %, under the proportional loop in case 4 and in manual in case 5, where the inlet has to be set by hand to hold the level once settled. Case 8, Actuator limit, opens the outlet wide, weakens the inlet valve to $c = 10\,\text{kg/s}$ per percent and moves the setpoint from 6 to 10 m. Case 9 runs the predictive controller of Section 2.5.5 through the outlet step to 50 % and a setpoint move to 11.2 m at 30 s, and is then repeated under the PID. Each case has a completion rule, as Section 1.6.4 describes.

Case 10, the challenge, applies the outlet step of case 4 with $K_c = 2\,\%/\text{m}$ and asks for the measured level within ± 1.5 m of 10 m over the last third of a 20 s run, from 13.2 s. It is judged on the case as given, so changing a plant constant or the outlet setting voids the attempt. With the shipped $K_c = 2\,\%/\text{m}$ the level is still falling, at 7.07 m, when the run ends, and the run fails with a worst error

of 2.97 m. The suggested $K_c = 10\,\%/\mathrm{m}$ with $K_I = K_D = 0$ passes with a worst error of 0.946 m, a margin of 0.55 m, and the app gives the same run whether the suggested tuning is entered before Run or carried through Reset. The band admits the 0.84 m proportional offset of case 4, and 0.10 m of the judged error is transmitter noise.

Case 2 runs open loop. The 1 % inlet step at $t = 5\,\mathrm{s}$ starts the level down at 0.05 m/s, the slope of Equation (2.7), and the level reaches 8.62 m at 35 s and 7.64 m at 60 s on its way to the 3.01 m resting level of the smaller inflow, with a time constant of 181 s at 10 m and a 10 % outlet. Cases 6 and 7 differ only in the proportional gain, and case 3 makes a larger move at the reference gain. Because each case starts with the bias that holds 10 m, a proportional loop is left short of a new setpoint only by the change in the required inlet position divided by K_c, and at $K_c = 4$, 10 and 20 %/m the level lies 0.033, 0.014 and 0.007 m below the stepped setpoint over 25 to 35 s (mean of the true level). The true level rises from 10 to 90 % of the move in 12.25, 4.5 and 2.0 s, read at the integration step, and over the 35 s run the scorecard valve travel of cases 6 and 7 is 0.39 and 2.06 times that of case 6 rerun at $K_c = 10\,\%/\mathrm{m}$, most of it driven by transmitter noise. Without noise the true level never reaches the setpoint, because a proportional loop on a first-order plant with no dead time does not overshoot. With the default noise it crosses the setpoint by 6 mm at $K_c = 10$ and 28 mm at $K_c = 20\,\%/\mathrm{m}$, and the overshoot tile of the scorecard, computed from the measured level, reads 9.3, 8.1 and 13.5 % for cases 3, 6 and 7, all of it noise. Case 4 parks 0.84 m low, as analyzed below. Case 5 starts from the same steady state in manual, so the outlet step drains the tank toward the 0.40 m resting level of the unchanged 2.21 % inlet, through 7.94 m at 10 s to 0.48 m at 60 s, unless the inlet is raised by hand, and 11.1 % holds 10.06 m. Case 8 starts at 6 m with the weaker inlet at 85.7 %. The setpoint move to 10 m drives the inlet valve to 100 % at the execution at $t = 5.0\,\mathrm{s}$, where it stays, and the level climbs toward 8.16 m, the most the valve can hold, reading 8.09 m at 60 s. No tuning closes the remaining 1.84 m.

> **TRY IT IN THE APP** **2 · Open-Loop Actuator Step**
>
> Load the case and press Run. At 5 s the inlet valve closes by one percentage point with the controller in manual. Watch the tank level trend: the level starts down at 0.05 m/s, passes 8.62 m at 35 s and 7.64 m at 60 s, and keeps falling toward the 3.01 m resting level of the smaller inflow.

The experiments of this section start from those cases and change the valve positions, the control mode, the three gains or the plant constants, as listed in Table 2.4. Every number below was computed from the logged history, which carries one row every 0.2 s, except that crossing times are read at the 0.05 s integration step and the valve statistics of Table 2.6 at every 0.1 s controller execution.

2.6.1 Manual Operation and the Steady-State Map

Equal increments of the inlet valve produce level steps that grow with the inventory already held, as shown in Figure 2.2. Five manual settings at 5 % intervals, each held for 50 s with the outlet wide open, end at levels of 0.51, 2.04, 4.55 and 8.01 m and then at the overflow limit, against the 0.51, 2.04, 4.59, 8.16 and 12.76 m resting levels of Equation (2.3). The fourth reading falls short of its prediction because 50 s is only 3.1 time constants at that level, and the fifth prediction lies above $h_{\max}$, so the tank spills instead of settling. The same run carries the time-constant result, because each successive approach takes longer than the one before it while the valve increments that caused them are identical.

2.6.2 Offset Under Proportional and Proportional-Integral Control

Table 2.5 reports the identical load disturbance, the outlet valve opened from 50 % to 70 %, applied to the same controller at two setpoints. Each run starts from case 4 with the transmitter noise set to zero and $K_I = 1\,\%/(\mathrm{m\,s})$ entered before the run, so that integral action finds the inlet position that holds the setpoint against the half-open outlet, which is the value listed in the *inlet before* row. At $t = 300\,\mathrm{s}$ the integral gain is set to the value of the column, which leaves the valve where it is, and

Table 2.4 Experiment Sequence and the Mechanism Each One Exposes

Experiment	Setup	Mechanism exposed
Fill and drain	case 1: inlet 100 % with the outlet shut, then inlet shut and outlet 100 %	$t_{\text{empty}} = 2A\sqrt{h_0}/c_v$ and the overflow limit
Step test	case 1, outlet 100 %: 1 % inlet step at three settled levels	$\hat{K}_u$ and $\hat{\tau}$ both scale as $\sqrt{\bar{h}}$
Outlet upset, P only	case 4 settled with $K_I = 1$, then $K_I = 0$ and outlet $50 \to 70$ %	offset proportional to the local process gain
Outlet upset, PI	same upset, $K_I = 1$	offset removed for under 1 % of added valve travel
Same tuning, two levels	repeat both at 3 m and 10 m	one tuning producing two closed-loop responses
Outlet closed	case 3 with $u_{\text{out}} = 0$ %, P only	proportional control with no offset on an integrator
Outlet open, carried bias	case 4 as shipped, outlet $10 \to 50$ %	offset $(u_{ss} - u_b)/K_c$ with the bias of the steady start
Noise and derivative	case 4, $K_I = 1$, K_D raised, $\sigma_n = 0.05$ m	valve activity from Equation (2.15)
Predictive control	cases 3, 4, 8, 9 and 10, MPC chosen before the run	offset removed through a disturbance estimate

Table 2.5 Response to an Outlet Valve Step From 50 % to 70 % With $K_c = 10\,\%/\text{m}$ and $K_D = 0$, $\sigma_n = 0$, Measured Over the 240 s Following the Upset

	$h_{\text{SP}} = 10$ m		$h_{\text{SP}} = 3$ m	
K_I (%/(m s))	0	1	0	1
K_u at the setpoint (m/%)	1.807	1.807	0.990	0.990
τ at the setpoint (s)	36.1	36.1	19.8	19.8
Level if uncontrolled (m)	5.10	5.10	1.53	1.53
Inlet before (%)	11.07	11.07	6.06	6.06
Inlet after (%)	15.17	15.50	8.18	8.49
Final offset (m)	0.411	0.000	0.212	0.000
Maximum deviation (m)	0.411	0.323	0.212	0.170
IAE (m s)	97.84	4.43	50.53	2.42
Valve travel (%)	4.1	5.0	2.1	2.5
Settling within 0.02 m of the final level (s)	5.50	30.10	4.05	26.20

the outlet is opened to 70 %. Every run therefore starts on setpoint with no inherited error whether or not it has integral action, and the offset reported is the deterministic one.

The two proportional columns differ only in how full the tank was. The same $K_c = 10\,\%/\text{m}$ leaves a standing error of 0.411 m at 10 m and 0.212 m at 3 m, a ratio of 1.94. The linearized prediction of that offset is $K_d\,\Delta u_{\text{out}}/(1 + K_u K_c)$ with both constants taken from Equation (2.6), which evaluates to 0.420 m and 0.220 m and to a ratio of 1.90, so the first-order model accounts for the measured offsets to within 2.1 % and 3.7 % even though the disturbance itself is far outside the range in which that model was derived. Without any control the same upset would take the level from 10 to 5.10 m and from 3 to 1.53 m by Equation (2.3), so proportional feedback alone recovers 92 % of the excursion at 10 m and 86 % of it at 3 m, and integral action recovers the rest. Figure 2.3 sets the offsets, errors and valve travel of the table side by side.

🕮 **Pause.** Try Exercise 2.2 before reading on.

Integral action carries almost no cost on this plant. Adding $K_I = 1\,\%/(\text{m s})$ removes the offset

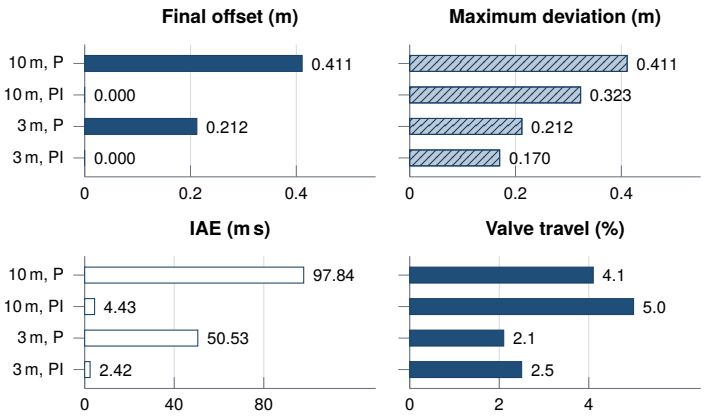


Figure 2.3 The outlet step of Table 2.5 at the two setpoints, proportional (P) and proportional-integral (PI) with the same K_c: final offset and valve travel (solid bars), largest deviation (hatched) and integrated absolute error (open). Notice that the same proportional gain leaves about twice the offset at 10 m as at 3 m, and that integral action removes both offsets for less than one percentage point of added valve travel.

entirely, cuts IAE from 97.84 m s to 4.43 m s at 10 m and from 50.53 m s to 2.42 m s at 3 m, reduces the maximum deviation in both cases, and spends 0.9 % and 0.4 % more valve travel. The trade that averaging level control is built around does not appear here, because the disturbance is a change in the discharge path itself rather than an inlet flow that has to be passed on. On this vessel the cost of tight regulation is measured in valve activity driven by the transmitter rather than in flow variation handed to a downstream unit, which is the subject of Section 2.6.4.

2.6.3 Proportional Control of the Integrating Configuration

Closing the outlet valve changes the character of the loop rather than its tuning, and it removes the offset rather than enlarging it. Setting $u = u_b + K_c e$ equal to the valve position u_{ss} the plant needs to hold still gives the steady-state error of a proportional loop directly, as shown in Equation (2.21), where u_b is the effective bias of Section 2.5.

$$e_\infty = \frac{u_{ss} - u_b}{K_c} \tag{2.21}$$

A vessel with no discharge holds still only with the inlet shut, so $u_{ss} = 0$. A loop that opens in automatic on the shut-outlet tank starts with the inlet shut, the only position that holds it still, so it carries $u_b = 0$, and the offset is therefore zero. Proportional action alone drives the level to setpoint and holds it there with $K_I = K_D = 0$, and it reaches the new setpoint after a setpoint move. Measured on case 3 with the outlet shut before the run, $K_c = 10\,\%/\mathrm{m}$ and no transmitter noise, the case starts at 10 m with the inlet shut, follows the scripted move to 11.2 m at $t = 5\,\mathrm{s}$, and holds the level within 1 mm of the new setpoint from $t = 18.85\,\mathrm{s}$ with the inlet valve shut again.

This property is the reason proportional-only control is accepted on integrating level loops, and it does not hold on the self-regulating side of the same vessel. Case 4 starts at 10 m with the outlet 10 % open and the inlet at the 2.21 % that holds it, which becomes the bias u_b. Once the outlet opens to 50 % at $t = 5\,\mathrm{s}$, holding 10 m requires 11.07 % of inlet valve, so Equation (2.21) predicts an offset of $(11.07 - 2.21)/K_c = 0.886\,\mathrm{m}$, and the loop settles 0.838 m low (mean of the true level from 60 to 300 s). The remaining discrepancy is accounted for by the square-root outflow law rather than by error in the prediction. At the depressed level of 9.162 m the required valve is only 10.59 %, which returns $(10.59 - 2.21)/K_c = 0.838\,\mathrm{m}$ exactly. No single bias is correct at every level and outlet setting

Table 2.6 Valve Activity of a Settled Loop at 10 m With the Outlet Half Open, $\sigma_n = 0.05\,\mathrm{m}$, $K_c = 10\,\%/\mathrm{m}$ and $K_I = 1\,\%/(\mathrm{m\,s})$, Measured Over 600 s

K_D (% s/m)	0	1	5
σ_u measured (%)	0.505	1.167	4.582
σ_u from Eq. (2.15) (%)	0.500	1.118	3.905
$\sigma_{\hat{h}}$ measured (m)	0.051	0.051	0.053
σ_h of the true level (m)	0.009	0.009	0.017
Valve travel (% per s)	5.8	15.8	66.3

on that side of the vessel, which is the direct argument for integral action, and it is why the runs of Table 2.5 reach their starting point with integral action on.

> **TRY IT IN THE APP** **4 · Disturbance Rejection**
>
> Load the case and run it for 60 s. When the outlet opens from 10 to 50 % at 5 s, the proportional loop opens the inlet but parks the level about 0.84 m below the 10 m setpoint. Then Reset, set the Integral gain to 1 before Run and run again: the level ends 0.001 m above the setpoint (Table 2.7).

Restoring the default 0.05 m of transmitter noise degrades the offset-free result, and the mechanism is the instrument rather than the controller. With the outlet shut the valve opens on any sample reading below setpoint, and the admitted liquid cannot be removed through a closed drain, so the level rises until such a reading has become rare. The level approaches 3.2 standard deviations above setpoint. In case 3 with the outlet shut it stands 0.12 m above the 11.2 m setpoint at five minutes and 0.16 m above it at fifty minutes with $\sigma_n = 0.05\,\mathrm{m}$, and 0.24 m and 0.32 m above it with $\sigma_n = 0.10\,\mathrm{m}$. Across five noise seeds the fifty-minute value spans 0.006 m and 0.012 m at the two noise levels. Opening the outlet removes the bias, because the vessel then discharges the excess.

Pause. Try Exercise 2.3 before reading on.

The bound itself is a property of keeping s_k unclipped. An implementation that clips the running sum instead discards the negative increments computed while the valve already sits at 0 % while applying the positive ones in full, and the resulting rectification leaves no resting point at all. The same experiment, run as a model counterfactual with the controller replaced by that variant, reached the 12 m overflow limit at $t = 30.1\,\mathrm{s}$, 25.1 s after the setpoint move, with no disturbance applied, and between 27.05 and 33.65 s across five noise seeds. The contrast makes the experiment a compact demonstration of why an integrating inventory with a one-way actuator is treated as a different control problem rather than as a slow first-order one.

2.6.4 Derivative Action and Measurement Noise

Table 2.6 reports the valve activity of a settled loop at 10 m with the default transmitter noise and three derivative settings, measured from case 4 with K_I and K_D entered before the run, over the 600 s from $t = 200\,\mathrm{s}$ with no further disturbance. The statistics are taken at every controller execution. The 0.2 s log gives the same standard deviations to within 2.5 % but only 2.8, 6.3 and 23.4 %/s of valve travel, because it sees every second valve move.

Equation (2.15) predicts the measured valve noise to within 1.0 % at $K_D = 0$ and 4.4 % at $K_D = 1\,\%\,\mathrm{s/m}$, and underpredicts it by 17 % at $K_D = 5\,\%\,\mathrm{s/m}$, because the expression keeps only the direct path from the transmitter to the valve. A valve deviation u' held for one execution moves the level by $c\,u'\Delta t_c/(\rho A) = 0.005\,u'$ in meters, and the derivative term returns that motion at the next execution as $-K_D c\,u'/(\rho A)$, which is $-0.25\,u'$ at $K_D = 5\,\%\,\mathrm{s/m}$. Adding that one-step autoregression to the noise path of Equation (2.15) raises the prediction to 4.50 %, within 2 % of the measurement. The true level is disturbed far less than the valve. Its standard deviation rises from 0.009 m, the residual of a loop that is only reacting to the instrument, to 0.017 m on the sampled integration phase, a third of the

Table 2.7 Shipped PID, Predictive Controller and a Proportional-Integral Reference on Five Cases With the Same Noise Seed

Case	Controller	IAE (m s)	IAE_h	$e_{\max}$ (m)	$\bar{e}_{\text{end}}$ (m)	Travel (%)	Limit (%)
3, setpoint	P	4.43	2.86	1.073	0.010	188.5	0
	PI[a]	4.98	3.47	1.072	−0.004	187.2	0
	MPC	3.07	0.98	0.916	−0.007	82.6	0
4, outlet	P	44.52	44.61	0.856	0.835	176.0	0
	PI[a]	9.65	8.89	0.660	−0.001	175.4	0
	MPC	5.87	4.16	0.649	−0.006	33.2	0
8, limit	P	133.61	133.68	3.969	1.918	29.2	91.7
	MPC	133.59	133.70	3.968	1.918	16.6	91.7
9, mismatch	P	48.42	48.51	1.928	0.899	191.1	0
	PI[a]	11.85	11.35	1.159	−0.015	190.1	0
	MPC	6.32	4.77	0.916	−0.006	91.6	0
10, challenge	P, $K_c = 2$	29.23	29.06	2.931	2.735	13.2	0
	P, $K_c = 10$[b]	11.28	11.12	0.846	0.837	61.3	0
	MPC	4.14	3.84	0.649	−0.005	17.6	0

IAE uses the measured level and IAE_h the true level. $e_{\max}$ is the largest true deviation after the first event, $\bar{e}_{\text{end}}$ the mean of $h_{\text{SP}} - h$ over the last 5 s, and Limit the share of the run with the valve at a limit. P is the shipped proportional-only tuning. [a]Reference, not shipped, $K_I = 1\,\%/(\text{m s})$. [b]Suggested, $K_I = K_D = 0$.

transmitter noise. Sampling the other integration phase gives about 0.012 m, so the apparent increase depends on the phase. Derivative action provides no compensating benefit on this plant. A first-order process with no dead time has no phase lag for derivative action to recover, so the only effect of K_D is to convert transmitter noise into valve travel at 66.3 % per s, which is 11 times the travel of the $K_D = 0$ case, and to increase the sampled variation of the true level. The near-doubling applies to the integration phase used in the table. That result is the reason derivative action is switched off in the large majority of industrial loops [15, 24], and the reason both K_I and K_D default to zero in the application.

> **+ DID YOU KNOW? Derivative Action That Only Moves the Valve**
>
> From the simulator: on the settled tank, raising the derivative gain from 0 to 5 % s/m raises the valve travel from 5.8 to 66.3 % per second, while the standard deviation of the true level grows only from 0.009 to 0.017 m (Table 2.6).

2.6.5 Comparison With the Predictive Controller

The controllers are compared on cases 3, 4, 8, 9 and 10 in Table 2.7, each case run for 60 s, or 20 s for the challenge, with the controller chosen before the run, so that both see the same events and the same noise sequence. The PID keeps the shipped tuning of each case and the MPC keeps the settings of Section 2.5.5 throughout. The shipped PID is proportional only, with $K_c = 10\,\%/\text{m}$ in cases 3, 4, 8 and 9, while the challenge ships $K_c = 2\,\%/\text{m}$ and suggests $K_c = 10\,\%/\text{m}$, both with $K_I = K_D = 0$, so the disturbance estimate gives the MPC an advantage of structure over every shipped tuning. A proportional-integral reference with $K_I = 1\,\%/(\text{m s})$, the setting of Table 2.5, is added on cases 3, 4 and 9 to separate the contribution of the model from that of integral action.

On the setpoint move of case 3, at the point where the model was fitted, every controller ends within 0.01 m of 11.2 m. The MPC opens the inlet to 31.7 % against 14.5 %, the true level rises from 10 to 90 % of the move in 0.85 s against 4.5 s with an overshoot of 0.041 m against 0.006 m, and the true-level IAE is 0.98 m s against 2.86 m s. Over the quiet first 5 s of cases 3, 4 and 9 the proportional loop moves the valve by 15.95 % and the MPC by 2.65 %, because the filter gain $\ell_x = 0.0569$ admits a small fraction of each reading and a plan is made only at every fifth execution, which accounts for most of the difference in travel.

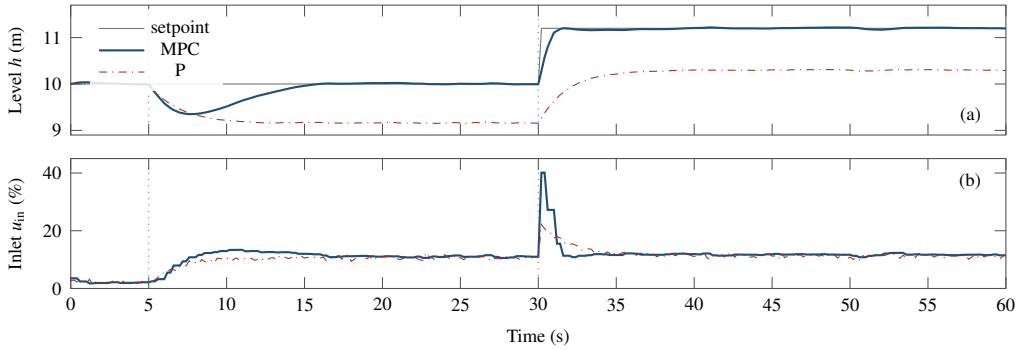


Figure 2.4 Case 9 under the predictive controller and under the shipped proportional loop, with the same noise seed. The outlet valve opens from 10 to 50 % at 5 s and the setpoint steps to 11.2 m at 30 s (dotted lines). (a) True level and setpoint. (b) Inlet valve position. Thick solid: MPC. Dash-dotted: shipped proportional loop. Thin gray: setpoint.

The outlet step of case 4 separates the structures. The proportional loop settles 0.835 m low, the offset of Equation (2.21), while the MPC ends 0.006 m above the setpoint with $\hat{d} = -8.79\,\%$, close to the −8.85 % that Equation (2.3) gives as $\bar{u}_m$ minus the 11.07 % that the half-open outlet requires. The proportional-integral reference also removes the offset, but with 8.89 m s of true-level IAE against 4.16 m s and 175.4 % of travel against 33.2 %. Most of the gain of the MPC over the shipped loop therefore comes from removing the offset, and the model with its filter adds a factor of 2.1 in IAE and 5.3 in travel.

Case 8 is a tie. Both controllers hold the inlet at 100 % from the execution at 5.0 s to the end, 91.7 % of the run, both leave the level at 8.094 m, and their IAEs differ by less than 0.02 m s, although the initial slope of the model is 5.0 times and its gain 64.5 times those of this tank with its weaker inlet. Every plan was optimal, and 110 of the 120 had the upper bound active, with up to 12 active-set iterations. The lower travel of the MPC comes from the 5 s before the step, 2.27 % against 15.76 %.

Case 9 joins the outlet step of case 4 and the setpoint move of case 3, so the move is made on a tank whose gain and time constant are a fifth of those of the model, as shown in Figure 2.4. The model keeps the initial slope of the vessel, so its response to a held 1 % inlet step after one decision period is 0.0249 m against 0.0248 m for the linearized tank, and the mismatch grows only with time, to 7.80 m against 1.81 m at the end of the horizon. Before the move $\hat{d}$ reaches −8.82 %. After it the MPC opens the inlet to 40.1 %, holds the true level within 0.1 m of 11.2 m from 1.0 s on and ends 0.006 m above the setpoint, while the proportional loop ends 0.899 m short, against the 0.902 m of Equation (2.21) at the depressed level, and the reference stays within 0.1 m from 4.05 s. The true-level IAE is 4.77, 48.51 and 11.35 m s and the travel 91.6, 191.1 and 190.1 % for the MPC, the shipped loop and the reference. A gain wrong by a factor of five costs the MPC little, because the estimator turns the mismatch into a load that the plan can see. Raising the move suppression from the shipped 0.03 to $\lambda = 1$, the top of its range, raises the IAE of the same run from 6.32 to 8.64 m s.

On the challenge the MPC repeats the first 20 s of its case 4 run, since it does not read K_c. Its worst measured error over the judged last third is 0.241 m, against 2.97 m for the shipped $K_c = 2\,\%/\mathrm{m}$, which fails, and 0.946 m for the suggested $K_c = 10\,\%/\mathrm{m}$, which passes but still ends 0.837 m low.

The MPC solved 120 plans in each 60 s case and 40 in the challenge, and none of the 520 was rejected. Every plan of cases 3, 4, 9 and 10 was the unconstrained optimum, so the advantage there comes from prediction and estimation rather than from constraint handling. The mean logged solve time was 0.3 to 0.6 ms per plan under Node.js and 0.54 to 0.64 ms in Chrome on case 9, at most 5.8 ms, or 0.11 to 0.13 % of one processor core in real time. These are wall-clock values on one desktop computer and vary between runs. The MPC loses none of the five cases and ties case 8, and

Table 2.8 Level Loops on Cases 3 and 4 Over a 180 s Run: Shipped Proportional, Counterfactual and Relay-Tuned PI, and the MPC, One Noise Seed

Case	Controller	IAE (m s)	IAE_h	e_{max} (m)	Travel (%)	$Travel^0$
3, setpoint	P, shipped	9.49	4.53	—	527.9	22.7
	PI[c]	9.92	4.49	—	526.0	23.0
	PI, relay	11.34	5.80	—	339.2	15.7
	MPC	8.06	2.16	—	123.3	60.0
4, outlet	P, shipped	144.7	145.1	0.860	515.4	8.4
	PI[c]	18.01	13.66	0.694	513.9	9.7
	PI, relay	18.93	13.93	0.916	333.3	10.7
	MPC	10.86	5.23	0.650	73.8	13.4

P, shipped: $K_c = 10$, $K_I = 0$. PI[c]: $K_c = 10$ with the relay's $K_I = 0.675$, a counterfactual that is neither shipped nor written. PI, relay: $K_c = 6.53$, $K_I = 0.675$. IAE uses the measured level, IAE_h the true level on the 0.05 s step, e_{max} is the largest true deviation after the outlet step, and $Travel^0$ repeats Travel with $\sigma_n = 0$. The two estimators give the same MPC rows.

most of its gain on cases 4, 9 and 10 is the integral action that the shipped tunings lack.

2.6.6 Estimator Choice and Relay Autotuning

These runs last 180 s, three trend windows, or 20 s for the challenge, with every setting entered before the run and the shipped noise seed, so their integrals exceed those of Table 2.7. Over the 1799 executions of case 9 under the MPC, the moving horizon and Kalman estimates differ by at most $6.1 \times 10^{-9}\sigma_n$ in the level and 3.2×10^{-9} % in $\hat{d}$. The residue is not rounding. The Riccati iteration that designed the filter stopped after 365 steps with its covariance a relative 5.4×10^{-10} short of the fixed point, which the window refines while the filter keeps its gain, and with the fixed-point covariance the difference is $8.9 \times 10^{-14}\sigma_n$. With each estimator in turn, the MPC runs of the ten cases give valve records within 5.1×10^{-9} % of each other, with no window rejected and no bound active. The Kalman estimate stays inside 0 to 12 m in every case, the lowest true level being the 0.40 m of the manual case 5, so no case separates the estimators and no trend of them is shown.

Against the true level, read only by the measuring script, both estimators follow case 3 with a root-mean-square error of 0.012 m and a largest error of 0.078 m, and case 9 with 0.070 m and 0.452 m. The largest error stays at or below 0.078 m in the five cases that keep the outlet setting and inlet coefficient of the fit, and reaches 0.45 to 0.61 m in the five that change either. An estimate took a median of 247 µs and a 99th percentile of 538 µs under Node.js. In Chrome, measured during case 9 on the animation loop that drives the application, the estimator and the planner together occupied at most 0.95 percent of a single core at normal speed and 7.3 percent at the tenfold speed, the higher reading of two runs taken at a load average of 2.5 to 3.1, which uses nearly three quarters of the budget of one tenth of a core.

The relay was started 6 s into case 3 with its setpoint step removed, so the tank held 10 m on the 2.21 % valve that holds that level. The relay was centered on $u_0 = 2.25$ %, and δ_r was limited to 2.03 %, 0.9 of the way to the closed valve. The experiment converged after 20 cycles and 48.0 s with $\alpha = 0.193$ m, $D = 0.55$, $P_u = 4.40$ s and $K_{cu} = 20.9$ %/m, and it wrote $K_c = 6.53$ %/m and $K_I = 0.675$ %/(m s) in place of the shipped 10 and 0. With $\varepsilon_r/\alpha = 0.776$, ψ is 51° and the result is flagged. Ten further noise seeds, 1001 to 1010, all converged, with K_{cu} from 14.9 to 51.1 %/m, P_u from 2.60 to 5.96 s and ψ from 42° to 72°. Started instead at 60 s of case 3, after the level has settled at 11.2 m, the relay converges in five cycles on $K_{cu} = 17.5$ %/m and $P_u = 3.41$ s with $\psi = 45°$, and it writes $K_c = 5.48$ %/m and $K_I = 0.732$ %/(m s), again a weaker proportional gain than the shipped one. Repeated after Reset with $\sigma_n = 0$, it returns 90.6 %/m at 5.80 s with $\psi = 79°$ and writes $K_c = 28.3$ %/m.

None of these values is the ultimate point of the tank. The model of Equation (2.16) is a lag with no dead time, and sampled with a zero-order hold at Δt_c its phase reaches $-180°$ only at the Nyquist frequency, where $K_{cu} = (1 + a)/b = 400$ %/m and $P_u = 2\Delta t_c = 0.2$ s. A relay with hysteresis cycles

where the imaginary part of the sampled frequency response G of the model equals $-\pi\varepsilon_r/(4\delta_r)$, and the same relay on the noise-free sampled model returns 107 %/m at 6.0 s, so the band places the cycle. Noise widens the swing, and at $\varepsilon_r/\alpha = 0.776$ Equation (2.20) turns a one percent error in α into a 2.5 percent error in K_{cu}, and the seeds spread K_{cu} threefold. The measured K_{cu} is 0.052 of the sampled ultimate gain and 0.195 of the noise-free relay result. The same mechanism separates the two runs started at 60 s: their swings differ by a factor of only 1.38, yet their gains differ by a factor of 5.2, because the noise-free swing lies just outside the band, where $\sqrt{\alpha^2 - \varepsilon_r^2}$ in Equation (2.20) is small.

The written constants improve only the outlet step in Table 2.8. On the setpoint step the true-level IAE rises from 4.53 to 5.80 m s, and the shipped K_c with the written K_I gives 4.49 m s, so the loss comes from the smaller K_c. On the outlet step the IAE falls from 145.1 to 13.93 m s, and the run ends 0.005 m from the setpoint where the shipped loop parks 0.841 m low. Integral action accounts for all of that gain, since the shipped K_c with the written K_I reaches 13.66 m s, and the smaller K_c raises the peak deviation from 0.694 to 0.916 m for 35 percent less valve travel. That travel is mostly transmitter noise passed through K_c, as Equation (2.15) predicts. It falls to 0.64 and 0.65 of the shipped loop's travel on the two cases, close to the ratio $6.53/10$ of the gains, and with $\sigma_n = 0$ it falls from 339 to 15.7 % on case 3 and from 333 to 10.7 % on case 4. On the outlet step the relay-tuned loop has 2.7 times the true-level IAE of the MPC, 13.93 against 5.23 m s, with 4.5 times its valve travel.

2.7 Technology in Practice

The loop of Section 2.3 reads LT through white noise and moves the inlet valve the instant it is commanded. Figure 2.5 draws it with the devices a plant tank would carry in the place of the schematic's symbols.

What Measures It

Three principles serve a vessel like TK-101. A hydrostatic transmitter near the floor reads the pressure of the liquid column. That pressure gives the height only while the density is known and constant, although in a vessel of constant area it measures the mass of liquid whatever its density [17]. A displacer weighs a body hanging in the liquid and cannot report a level above the top of that body [10]. An echo transmitter times a wave reflected from the surface [17]. No range, accuracy or response time is given here, because none was found in an open source that names no maker. The noise σ_n of Equation (2.4) stands for the turbulence of the surface. The model's transmitter has no density error, no span limit and no fouling, so $\hat{h}$ and h differ only by that noise.

> **TECHNOLOGY** **Sensor: A Guided-Wave Radar Level Transmitter**
>
> A guided-wave radar transmitter sends microwave pulses down a metal probe that hangs into the liquid and times the echo from the surface. The wave travels through the vapor above the liquid, so the calibration rests on the speed of light in that vapor, which shifts when the vapor's pressure or temperature changes. A probe may be a single rod, a pair of rods or a rod inside a tube. The coaxial form guides the energy best, and the single rod tolerates fouling best. Sticky deposits on a probe can return echoes that the transmitter takes for the liquid surface, so the reading reports the deposit as the level [17].

What Moves It

The inlet valve of Equation (2.1) is linear and has no friction. A plant inlet is a throttling valve with a pneumatic actuator, usually fitted with a positioner that corrects the difference between the commanded and the actual stem position [17]. Its failure position follows from the hazard, not from habit: an air-to-open valve fails closed, and the risks of the process should decide which way it fails [17]. On a tank that can overfill, a valve that shuts when its air fails stops the inflow. The outlet of TK-101 is a hand valve, the disturbance of every case.

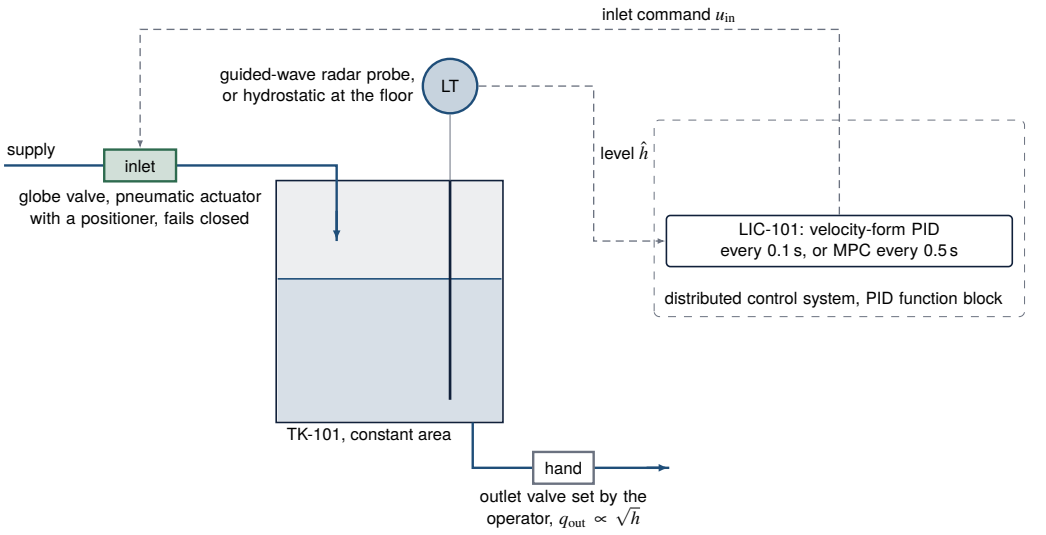


Figure 2.5 The level loop of the chapter with plant devices in place of the schematic's symbols. LT reads the level from the roof with a guided-wave radar probe, or from the floor by hydrostatic pressure, and LIC-101 on the control platform moves the inlet valve through its positioner. Notice that the outlet is a hand valve the loop never moves, so with the outlet shut the loop can add liquid but has no way to remove it.

> **TECHNOLOGY** **Final Element: A Globe Valve and Its Stem Friction**
>
> A globe valve is built for fine throttling and is widely used to regulate flow [25]. Packing around its stem adds friction, and friction at rest exceeds friction in motion. The stem of a pneumatic valve therefore sticks until the actuator pressure is high enough to break it free, jumps, and sticks again, so a smooth command becomes a staircase of stem positions. A reversal must first relax the friction in one direction and then overcome it in the other, which acts as a dead band [17]. On a trend, a sticky valve shows a sawtooth controller output and a process variable that cycles around its setpoint [25].

Where the Controller Runs

A distributed control system runs production at one site in refining, chemical manufacturing, water treatment and power generation. Its loops run with cycle times from milliseconds to minutes [26]. LIC-101 executes every 0.1 s and the predictive controller plans every 0.5 s, both inside that range. The velocity-form PID of Equation (2.13) is the PID function block of such a platform, with the manual station and the bumpless transfer of Section 2.5.

> **IN PRACTICE** **Test the Valve Before Tuning the Loop**
>
> An instrument handbook advises a test of the valve before any tuning. With the controller in manual, the output is stepped up and down by small amounts, and a valve with excess friction shows up as a process variable that fails to move after a reversal. Valves with poor maintenance, or without a positioner to correct the stem, are the worst. Tuning cannot remove the cycling that friction causes: a detuned loop cycles more slowly but still cycles. An integral dead band stops the cycling at the price of a standing error no larger than the band [17].

What Is Modernizing It

Three developments touch this loop. The first is the echo instrument. Some industries replace old displacer level instruments with guided-wave radar, even at high pressure, and a common retrofit

installs the radar probe in the cage that once held the displacer [17]. The second is the smart valve. Many modern refineries use valves with position feedback, self-diagnostics and health monitoring, although most valves still lack them [25]. The third is data-driven diagnosis of the loop from the controller output and the process variable alone, the subject of the open challenge at the end of Section 2.9 [27].

2.8 Limitations

The simulator is a teaching instrument and not a design tool. A single well-mixed inventory with a quasi-steady discharge cannot represent outlet line inertia, surface turbulence, vortexing at the drain or any two-phase behavior, so it must not be used to size a drain or a surge volume. The vessel is a vertical cylinder of constant area, which removes the geometric nonlinearity that a horizontal or a conical vessel adds on top of the hydraulic one. The actuators are ideal, so stiction, hysteresis and stroking time are absent, and the limit cycle they cause in industrial level loops cannot be produced. The transmitter carries white noise and nothing else, so the low-frequency errors that dominate a real level measurement are not represented. Mass is not conserved during an overflow. There is no cascade to a flow controller, no second tank, and no way to make the outlet flow independent of the level other than closing the valve entirely, so the pumped-outflow case is available only as the zero-flow limit rather than as a nonzero fixed discharge. Two gravity-drained tanks in series are the plant of Chapter 3. All ten shipped cases start at steady operation, so a start-up from an empty tank has to be set up by hand, by shutting the inlet and opening the outlet before the run as in Figure 2.2. The noise source is reseeded whenever a case is loaded or reset, so the noise realization is the same for every user, and a result should not be treated as an average over realizations. Each simplification that was retained, the square-root discharge, the two hard limits, the level-dependent gain and the short control period, was retained because removing it removes a behavior the loop must be tuned around. Section 2.3.4 states where each assumption fails.

The predictive controller plans on a first-order pair fixed at 10 m with the outlet 10 % open. It leaves out the square-root outflow, so its gain and time constant are five times too large with the outlet half open, and it leaves out the outlet valve, the overflow and the dry limit, so no plan protects the rim. Its Kalman gain assumes the default 0.05 m of noise and is not recomputed when σ_n or a plant constant changes, its λ sits at the lowest value of the shared tuning sweep, and its comparison uses one noise realization and excludes the shut-outlet tank, an integrator that the self-regulating model does not describe. The bounds of the moving horizon estimator apply to the states of the same first-order model, so where the model is wrong the estimate is wrong inside them, and no case on this tank activates them. The relay result describes the experiment as much as the vessel, because the phase of a delay-free lag approaches -180° only through the sampling, and the Tyreus–Luyben rule is one of several published conversions.

2.9 Discussion

Level loops carry a share of plant safety that is large relative to the attention they receive. The two disturbances the application offers, a drain that opens and a drain that closes, bracket a real hazard. A drain that closes while the inlet keeps filling is an overfill, and an overfill of a storage tank produced the vapor cloud at Buncefield, where the release rate and the cascade of liquid down the tank wall are what generated the flammable inventory rather than any subsequent process upset [8]. The refinery tower of the history box in Section 2.2 was overfilled while its level indication read within its span, and a dynamic reconstruction of the sequence shows how far the true inventory had diverged from the instrument before anything else went wrong [28]. Both events turn on a distinction the application makes visible in its simplest form, that the level a controller acts on is $\hat{h}$ and the level that overflows the tank is h.

The opposite requirement is just as real and pulls the other way. A surge vessel is installed to absorb variation, and a level controller tuned for tight regulation converts an inlet flow disturbance

into an outlet flow disturbance of the same size and hands it to the next unit. Averaging level control exists to spend the available tank volume instead, and the tuning parameter of a well-posed averaging controller is the level swing that is allowed rather than a gain [1, 12]. Consistency of the inventory loops across a plant, in the sense that each holdup is regulated by a flow in a direction compatible with how throughput is set, is a structural requirement that no amount of local tuning can supply [13]. The shut-outlet experiment of Table 2.4 is the smallest instance of that structural problem, in which one inventory has no flow available to regulate it in the required direction.

> **OPEN CHALLENGE** **Finding the Sticky Valves in a Plant**
>
> Valve stiction is a common fault in process plants, and it brings oscillation, actuator wear and higher maintenance cost. Many plants still run conventional valves without position feedback or self-diagnostics, so the only signals available are the controller output and the process variable, and detection rests on manual inspection and operator judgment [25]. Detectors trained on simulated loops often fail on real ones, because simulation cannot fully reproduce plant data [27]. Progress would be a detector that works on any loop from those two routine signals and warns well before the valve fails, which one study on refinery data has begun to show, predicting stiction up to four hours ahead [25].

2.10 Conclusion

This chapter studied a gravity-drained tank as an interactive laboratory in which the plant model is a function of the inventory. The simulator integrates one state under a mass balance whose discharge follows the Torricelli square-root law, with a linear inlet valve, hard limits at the overflow height and at the dry bottom, and a transmitter carrying 0.05 m of Gaussian noise. Linearization gives a first-order model whose gain and time constant both scale as the square root of the operating level while their ratio stays fixed at 0.05 m/s per percent of inlet travel, and step tests in the running application confirm the scaling with an identified gain rising by a factor of 2.14 and an identified time constant by 2.12 between 2.04 m and 9.88 m against a predicted factor of 2.20. The identified time constant equals the drain time of the same vessel, measured at 19.65 s from a full tank against the 19.79 s of the analytic expression.

The same outlet upset applied at two setpoints leaves a proportional loop 0.411 m short at 10 m and 0.212 m short at 3 m with identical gains, a ratio of 1.94 against the 1.90 of the linearized prediction, and integral action removes both at the cost of 0.9 % and 0.4 % of extra valve travel. The default controller is a velocity-form PID with derivative on the measurement, whose integral alone is corrected at the valve limits by back-calculation and whose manual tracking fixes the effective bias of a proportional loop at the moment of transfer. Its short 0.1 s sample period makes a derivative gain of 1 % s/m contribute as much valve noise as the entire default proportional gain.

A linear predictive controller fitted at 10 m with the outlet 10 % open removes the offset through its disturbance estimate when the outlet opens to 50 %, ending case 9 within 0.01 m of its new setpoint with a true-level IAE of 4.77 m s against 48.51 m s and less than half of the valve travel of the shipped proportional loop, which ends 0.899 m short. A moving horizon estimator offered in place of its filter gives valve records within 5.1×10^{-9} % of the filter's on all ten cases, because no bound becomes active, and a relay autotune lowers the true-level IAE of the outlet step from 145.1 to 13.9 m s through the integral action it adds.

Closing the outlet valve turns the same vessel into a pure integrator that no inlet valve can bring back down, which places both textbook level models in one apparatus. A fuller model would add a fixed-rate pumped discharge as a third configuration, a nonconstant vessel cross-section, valve stiction, a predictive controller that spends the surge volume against the overflow constraint, and a second tank in series so that averaging control has somewhere to send the disturbance. The tank is an accessible platform for level experiments in which the tuning the loop requires depends on the inventory the vessel holds.

KEY POINTS

- A tank that drains under its own head settles at a level of its own, and a fuller tank answers the same valve move more strongly and more slowly.
- Gain and time constant both grow as the square root of the level, and their ratio, the initial slope of every inlet step, stays at 0.05 m/s per percent.
- The same proportional loop leaves 0.411 m of offset at 10 m and 0.212 m at 3 m, and integral action removes both.
- With the outlet shut the tank is an integrator: a proportional loop has no offset, and no inlet valve can bring the level back down.

Where else this applies. Volume 2, Chapter 2 has the same fixed ratio of gain to time constant, and Chapter 9 moves between the same two characters.

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2.11 Exercises

2.1 Concept check · 2 min

The inlet valve of the default tank is stepped up by 1 % three times: from a settled 2 m with the outlet wide open, from a settled 10 m with the outlet wide open, and from a settled 10 m with the outlet 10 % open. How does the rate at which the level starts to rise compare across the three tests?

(a) It is the same in all three, and only the final rise and the time to reach it differ

(b) It grows with the square root of the level, so it is largest in the two tests at 10 m

(c) It is largest with the outlet 10 % open, because the process gain is largest there

(d) It is the same at both levels with the outlet wide open, and smaller with the outlet nearly shut

2.2 Concept check · 2 min

The same proportional loop, $K_c = 10\,\%/\mathrm{m}$ with $K_I = K_D = 0$, holds the tank on setpoint at 3 m in one run and at 10 m in another, both with the outlet 50 % open. The outlet then opens to 70 %. How do the two standing errors compare?

(a) They are equal, because the controller gain and the outlet step are the same

(b) The error at 10 m is about twice that at 3 m, as K_d grows faster than K_u

(c) The error at 3 m is the larger one, because the loop gain $K_u K_c$ is smaller there

(d) Neither run keeps an error, because the drain makes the tank regulate itself

2.3 Concept check · 2 min

Case 3 is run with the outlet valve shut before Run, a proportional-only loop at $K_c = 10\,\%/\mathrm{m}$ and the default 0.05 m of transmitter noise. The setpoint moves to 11.2 m at 5 s. Where does the true level stand five minutes into the run?

(a) On the setpoint, because a proportional loop on an integrator has no offset

(b) A little below the setpoint, because the valve is shut for part of every second

(c) About 0.12 m above it, because low readings open the valve and nothing drains

(d) At the 12 m rim, because the noise winds up the running sum of the PID

2.4 Theory and calculation · 8 min

The default tank has $A = 1\,\mathrm{m}^2$, an inlet coefficient $c = 50\,\mathrm{kg/s}$ per percent, $\rho = 1000\,\mathrm{kg/m}^3$ and an outlet coefficient $c_v = 0.35\,\mathrm{m}^3/\mathrm{s}$ per $\mathrm{m}^{1/2}$. The outlet is 50 % open.

(a) Find the inlet position that holds the level at 3 m. *Checkpoint:* 6.06 %.

(b) Find the time constant, the process gain and the disturbance gain of the linearized model at that point. *Checkpoint:* 19.8 s, 0.990 m/% and −0.12 m/% of outlet travel.

(c) A proportional loop with $K_c = 10\,\%/\mathrm{m}$ holds the level at 3 m when the outlet opens to 70 %. Predict the standing error, and the level the tank would reach with the inlet left where it was. *Checkpoint:* 0.220 m, and 1.53 m.

2.5 Theory and calculation · 5 min

The default tank ($A = 1\,\mathrm{m}^2$, $c_v = 0.35\,\mathrm{m}^3/\mathrm{s}$ per $\mathrm{m}^{1/2}$) is full at 12 m. The inlet is shut and the outlet opened wide.

(a) Integrate Equation (2.1) to find the time to empty the tank, and compare it with the time constant of Equation (2.6) at the starting level. *Checkpoint:* 19.79 s for both.

(b) Find the times at which the level passes 9, 6 and 3 m. *Checkpoint:* 2.65, 5.80 and 9.90 s.

2.6 Theory and calculation · 7 min

Case 4 starts at 10 m with the outlet 10 % open and the inlet at the 2.214 % that holds it, under a proportional loop with $K_c = 10\,\%/\mathrm{m}$ and no integral action. At 5 s the outlet opens to 50 %, where holding 10 m needs 11.068 %.

(a) Predict the offset with Equation (2.21), taking the bias as the starting inlet position. *Checkpoint:* 0.885 m.

(b) The loop settles below 10 m, where less inlet is needed. With the outlet half open the inlet that holds a level h is $3.5\sqrt{h}$ percent. Find the level at which the proportional loop supplies exactly that inlet. *Checkpoint:* 9.162 m, an offset of 0.838 m.

(c) The same loop runs case 3 with the outlet shut before Run, and nothing opens it. Find the offset with no transmitter noise. *Checkpoint:* Zero.

2.7 App challenge · 7 min

Load scenario 2, Open-loop actuator step, and change no setting. The case starts in Manual at 10 m with the outlet 10 % open, and the inlet steps from 2.214 to 1.214 % at 5 s. Run to 35 s at the default seed and export the CSV file.

(a) From the true level, column `true_pv`, in the rows at 5.0 and 6.0 s, estimate the rate at which the level starts to fall. *Checkpoint:* 0.050 m/s, within 0.002 m/s.

(b) Read the true level at 35 s, and compare it with the linearized model of Equation (2.5) at this point. *Checkpoint:* 8.62 m, within 0.01 m, for both.

(c) Find the level at which the tank will settle, and the final level the linearized model predicts. *Checkpoint:* 3.01 m against 0.96 m.

2.8 App challenge · 8 min

Load scenario 4, Disturbance rejection. The outlet opens from 10 to 50 % at 5 s. Use the default seed, run each case for 60 s and export the CSV file of each.

(a) Run the scenario as shipped, a proportional loop with $K_c = 10\,\%/\mathrm{m}$. Read the IAE and the MV travel tiles of the scorecard, and find the mean true level over the last 5 s. *Checkpoint:* IAE 44.52 m s and travel 176.0 %, each within 0.2. Mean level 9.165 m, within 0.01 m.

(b) Press the reload button, set the Integral gain to 1 %/(m s) and run again. Read the same three values. *Checkpoint:* IAE 9.65 m s, travel 175.4 %, mean level 10.00 m, within 0.01 m.

Compare the offset of run (a) with the prediction of Equation (2.21), and say what the integral action cost.

2.9 App challenge · 9 min

Load scenario 9, MPC on a tank it was not fitted to. The outlet opens from 10 to 50 % at 5 s and the setpoint moves to 11.2 m at 30 s. Use the default seed and run each case for 60 s. The exported file begins with comment lines marked `#`.

(a) Run the scenario as shipped, with the Controller at MPC. Read the IAE tile, and the disturbance estimate, column `est_d`, in the row at 29.8 s. *Checkpoint:* 6.32 m s and −8.82 %.

(b) Set the Controller to PID, press Reset and run again. Read the IAE tile and the mean true level over the last 5 s. *Checkpoint:* 48.42 m s and 10.30 m.

(c) Set the Controller back to MPC, set the Move suppression (Advanced level) to 1, press Reset and run again. Read the IAE tile. *Checkpoint:* 8.64 m s.

Accept an IAE within 0.1 m s, an estimate within 0.1 % and a level within 0.01 m. Explain how a model five times too sensitive still ends without offset.

2.10 Application area · 4 min

The default outlet has $C_d A_o = 0.0790\,\mathrm{m}^2$, which is 7.9 % of the tank cross-section. A plant tank of the same $1\,\mathrm{m}^2$ drains through an orifice of 50 mm diameter with $C_d = 0.6$.

(a) Find its outlet coefficient c_v with Equation (2.2), taking $g = 9.81\,\mathrm{m/s^2}$. *Checkpoint:* $0.005\,22\,\mathrm{m^3/s}$ per $\mathrm{m}^{1/2}$.

(b) Find its time constant at 10 m with the outlet wide open, and compare it with that of the default tank. *Checkpoint:* About 1210 s, or 20 min, against 18.1 s.

2.11 Application area · 3 min

A surge drum feeds a downstream column, and a manager asks for its level loop to be tightened, citing Table 2.5: integral action removed the offset for under 1 % of extra valve travel. Give two reasons from this chapter why that table does not support the request, and name what the app would need in order to show the trade.

2.12 Review the agent · 5 min

An AI assistant is asked how to tighten the level loop of case 4, which runs with $K_c = 10\,\%/\mathrm{m}$ and $K_I = 1\,\%/(\mathrm{m\,s})$, and replies as follows. “Add derivative action, $K_D = 5\,\%\,\mathrm{s/m}$. Derivative action anticipates the error and adds phase lead, and the controller executes every 0.1 s, so the derivative always sees fresh data. The level will stay closer to its setpoint for about the same valve movement.” Find the flaw in the proposal and say how it could be checked.