Application of KKT Conditions

For a problem in the following form,

\[
\begin{align*}
\text{Min} & \quad f(x) \\
\text{s.t.} & \quad g_i(x) - b_i \geq 0 \quad i = 1, \ldots, k \\
& \quad g_i(x) - b_i = 0 \quad i = k+1, \ldots, m
\end{align*}
\]

A) Give the KT necessary conditions, explaining each equation.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_i(x^*) - b_i ) is feasible ( i = 1, \ldots, m )</td>
<td>Primal Feasibility</td>
</tr>
<tr>
<td>( \nabla f(x^<em>) - \sum_{i=1}^{m} \lambda_i^</em> \nabla g_i(x^*) = 0 )</td>
<td>Dual Feasibility</td>
</tr>
<tr>
<td>( \lambda_i^* \left[ g_i(x^*) - b_i \right] = 0 \quad i = 1, \ldots, k )</td>
<td>Complementary slackness</td>
</tr>
<tr>
<td>( \lambda_i^* \geq 0 \quad i = 1, \ldots, k )</td>
<td>Positive Lagrange multipliers</td>
</tr>
</tbody>
</table>

B) A cylindrical storage tank is to be constructed for which the following costs apply:

- Metal for sides: $30.00/sq. ft.
- Concrete base and metal bottom: $37.50/sq. ft.
- Top: $7.50/sq. ft.

The tank is to be constructed with dimensions such that the cost is a minimum for whatever capacity is selected. One possible approach to selecting the capacity is to build the tank such that an additional cubic foot of capacity costs $8. (Note this does not mean $8 per cubic foot average for the entire tank.) Find the optimal diameter and height of the tank.

\[
\begin{align*}
\text{dollars} & \quad := 1 \\
c_{\text{side}} & \quad := 30 \frac{\text{dollars}}{\text{ft}^2} \\
c_{\text{base}} & \quad := 37.50 \frac{\text{dollars}}{\text{ft}^2} \\
c_{\text{top}} & \quad := 7.50 \frac{\text{dollars}}{\text{ft}^2} \\
d & \quad = \text{Diameter} \\
h & \quad = \text{height} \\
V & \quad = \text{capacity} \\
A_{\text{side}} & \quad = \pi \cdot d \cdot h \\
A_{\text{base}} & \quad = A_{\text{top}} = \frac{\pi}{4} \cdot d^2
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad \text{total cost} \\
\text{s.t.} & \quad \text{Volume Equation}
\end{align*}
\]
Write the KT Conditions

1. Primal Feasibility
\[ \frac{\pi}{4} \cdot d^2 \cdot h - V = 0 \]

2. Dual Feasibility
\[
\begin{align*}
& c_{\text{side}} \cdot \pi \cdot h + \left( c_{\text{top}} + c_{\text{base}} \right) \cdot \frac{\pi}{2} \cdot d - \lambda_1 \cdot \left( \frac{\pi}{2} \cdot d \cdot h \right) = 0 \\
& c_{\text{side}} \cdot \pi \cdot d - \lambda_1 \cdot \left( \frac{\pi}{4} \cdot d^2 \right) = 0
\end{align*}
\]

3. Complementarity slackness and 4. Positive Lagrange multipliers don't apply

Solve KT Conditions

Start with last equation in dual feasibility section

\[ c_{\text{side}} \cdot \pi \cdot d - \lambda_1 \cdot \frac{\pi}{4} \cdot d^2 = 0 \quad \text{Eliminate } d=0 \text{ solution} \]

Solve for \( d \)
\[ d = \frac{4c_{\text{side}}}{\lambda_1} \]

\[ d = 15 \, \text{ft} \]

Use first dual feasibility equation to solve for \( h \)
\[
\begin{align*}
& c_{\text{side}} \cdot \pi \cdot h + \left( c_{\text{top}} + c_{\text{base}} \right) \cdot \frac{\pi}{2} \cdot d - \lambda_1 \cdot \frac{\pi}{2} \cdot d \cdot h = 0 \\
& h = \frac{-\left( c_{\text{top}} + c_{\text{base}} \right) \cdot \frac{\pi}{2} \cdot d}{c_{\text{side}} \cdot \pi - \lambda_1 \cdot \frac{\pi}{2} \cdot d} \quad h = \frac{-\left( c_{\text{top}} + c_{\text{base}} \right) \cdot d}{2c_{\text{side}} - \lambda_1 \cdot d}
\end{align*}
\]

\[ h = 11.25 \, \text{ft} \]

Use Primal Feasibility Constraint to solve for \( V \)
\[
\begin{align*}
& V = \frac{\pi}{4} \cdot d^2 \cdot h \\
& V = 1988 \, \text{ft}^3
\end{align*}
\]