CHAPTER 8
ROBUST DESIGN

1 Introduction

In the "real" world, almost all designs are subject to variation. Such variation can arise from multiple sources, including manufacturing processes, material properties, changing operating conditions, or the environment. We can also have uncertainty associated with our computer model. We may not know some assumed values as well as we would like (e.g. heat transfer coefficients, friction coefficients), and our assumptions about boundary conditions might also be faulty. For example, loads or temperatures might be different than we assumed.

The consequences of variation are almost always bad. Variation in product dimensions can lead to assemblies which assemble poorly or not at all, or function improperly. Failure to take into account variation can lead to product failure, poor performance and customer dissatisfaction. A famous quality researcher, Genichi Taguchi, has promoted the idea that any deviation from a desired target value results in a loss to the customer.

Optimized designs may be particularly vulnerable to variation. This is because optimized designs often include active or binding constraints. Such constraints are on the verge of being violated. Slight variations in problem parameters can cause designs to become infeasible.

Thus it should be clear that we should not only be interested in an optimal design, but also in an optimal design which is *robust*. A robust design is a design which can tolerate variation. Fortuitously, a general approach to robust design can be formulated in terms of optimization techniques, further extending the usefulness of these methods. In this chapter we will learn how to apply optimization methods to determine a robust design.

We will define variation in terms of tolerances which give upper and lower limits on the expected deviation of uncertain quantities about their nominal values. We consider a design to be robust if it can tolerate variability, within the ranges of the tolerances, and still function properly. The term "function properly" will be taken to mean the constraints remain feasible when subjected to variation. We define this type of robustness as *feasibility robustness*.

2 Worst-case Tolerances

2.1 Introduction

We will begin by considering *worst-case* tolerances. With a worst-case tolerance analysis, we assume all tolerances can simultaneously be at the values which cause the largest variation. We ignore the sign of the variation, assuming it always adds. This gives us a conservative estimate of the worst situation we should encounter.

2.2 Background

We will consider a design problem of the form,

$$\begin{array}{ll} \text{Min} & f(\mathbf{x},\ \mathbf{p}) \\ \text{s.t.} & g_i(\mathbf{x},\ \mathbf{p}) \le b_i \quad i = 1,\ldots,m \end{array}$$

where

x is an *n* dimensional vector of design variables
p is a *l* dimensional vector of constant *parameters*, i.e., unmapped analysis variables.

We will group the right-hand-side values, b_i, into a vector **b.**

For a given set of nominal values for **x**, **p**, and **b**, there can be fluctuations $\Delta\mathbf{x}$, $\Delta\mathbf{p}$, and $\Delta\mathbf{b}$ about these nominal values. We would like the design to be feasible even if these fluctuations occur. As we will see, in a constrained design space, *the effect of variation is to reduce the size of the feasible region.*

2.3 Two Approaches to Robust Optimal Design

Several researchers have incorporated worst-case tolerances into the design process, using a "tolerance box" approach, as illustrated in Fig. 8.1. A tolerance box is defined for the design variables; the robust optimum is the design that is as close to the nominal optimum as possible and keeps the entire box in the feasible region. A main drawback is that it does not allow us to specify tolerances on parameters.

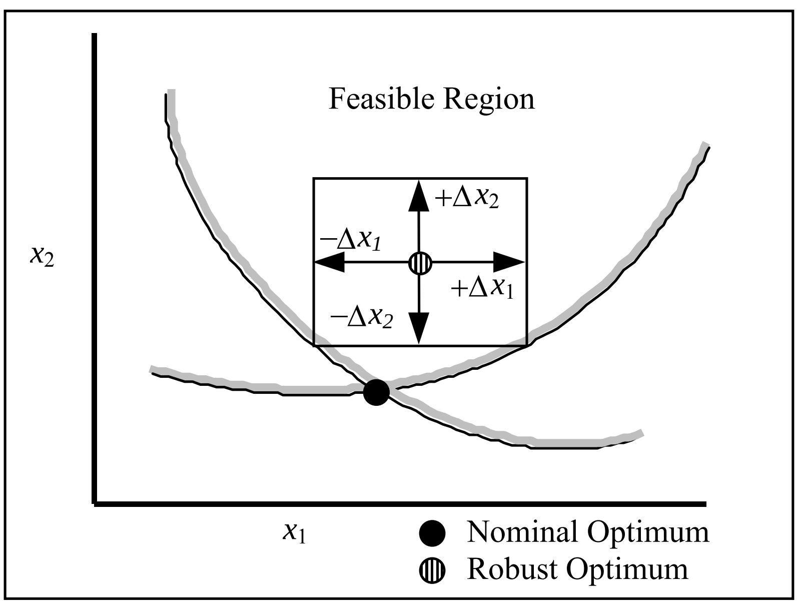


Fig 8.1. Tolerance box approach for robust design with worst-case tolerances.

In contrast to the tolerance box approach, the method we will develop relies on "transmitted variation." As will be explained, we *transmit the variation from the variables and parameters to the constraints*, and then correct the nominal optimum so that it is feasible with respect to the constraints with the transmitted variation added in. This method is illustrated in Fig. 8.2. The same optimization methods used to find the nominal optimum can be used to find the robust optimum, and tolerances may be placed on any model value, whether a variable or a parameter.

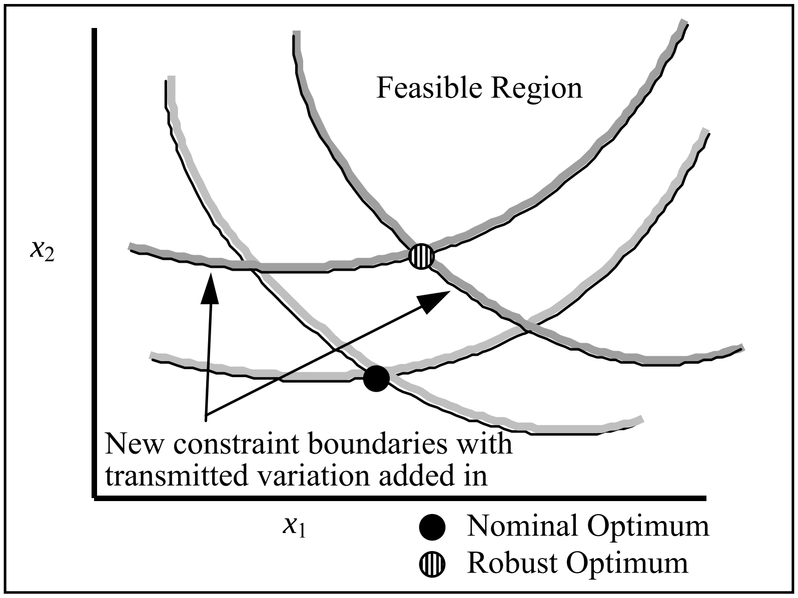


Fig 8.2. Transmitted variation approach for robust design with worst-case tolerances.

2.4 Calculating Transmitted Variation: Linear Analysis

Worst-case tolerance analysis assumes that all fluctuations may occur simultaneously in the worst possible combination. The effect of variations on a function can be estimated from a first order Taylor series, as follows:

$$\Delta g_i = \sum_{j=1}^{n} \left| \frac{\partial g_i}{\partial x_j} \Delta x_j \right| + \sum_{j=1}^{m} \left| \frac{\partial g_i}{\partial p_j} \Delta p_j \right| \tag{8.1}$$

where the bars indicate that the absolute value is taken. With the absolute value, (8.1) allows the tolerances to assume any sign and therefore computes the worst possible effect of the tolerances. We will refer to Δg_i as the “functional variation.” For constraints, we must also add in variation of the right hand side Δb_i,

$$\Delta_i = \Delta g_i + \Delta b_i \tag{8.2}$$

We will refer to Δ_i as the “total constraint variation.”

2.5 Developing a Robust Optimal Design

2.5.1 Compensating for Variation

Robustness can be developed for worst-case tolerances by adjusting the value of the constraint functions by the amount of the total constraint variation during optimization. For a less than constraint, we add the variation; for a greater than constraint, we subtract the variation. In both cases, this has the effect of reducing the size of the feasible region, with a corresponding degradation in the value of the objective. Thus a less than constraint becomes,

$$g_i + \Delta_i \le b_i \tag{8.3}$$

A greater than constraint becomes,

$$g_i - \Delta_i \ge b_i \tag{8.4}$$

Alternatively, we can consider that the variation has reduced (or increased) the right side, depending on whether we have a less than or greater than constraint, respectively:

$$g_i \le b_i - \Delta_i \tag{8.5}$$

$$g_i \ge b_i + \Delta_i \tag{8.6}$$

2.5.2 An Efficient Solution Method

Adding in the transmitted variation can be computationally expensive because the transmitted variation is a function of derivatives, and these would have to be evaluated every time the constraint is evaluated.

To reduce computation, we propose the following process,

1. Drive to the nominal optimum.
2. Calculate the transmitted variation.
3. Adjust the constraint right hand sides by the amount of transmitted variation.
4. Assuming the transmitted variation is a constant, re-optimize to find the robust optimum.

The assumptions that are built into this method are,

- the robust optimum is close to the nominal optimum.
- the derivatives are constant, i.e. second derivatives are equal to zero.

These assumptions are consistent with assuming a linear Taylor expansion, (8.1), for the transmitted variation.

2.5.3 *An Example: The Two-bar Truss*

We will illustrate the method on our familiar example, the Two-bar Truss. Now, however, we will add in tolerances on all the analysis variables and the right hand side for stress. (The right hand side for stress is the yield strength, a material property, and so has variation associated with it. The other right hand sides are set by the user and are not uncertain.) Data regarding the truss are given in Table 1.

Table 1 Worst-case Tolerance Data for the Two-bar Truss

Description	**Nominal Value**	**Worst-case Tolerance**
Height, H	Design Variable	0.5 in
Width, B	60 in	0.5 in
Diameter, d	Design Variable	0.1 in
Thickness, t	0.15 in	0.01 in
Modulus	30000 ksi	1500 ksi
Density	0.3 lb/in^3	0.01 lb/in^3
Load, P	66 kips	3 kips
Yield Strength *Right Hand Side*	100 ksi	5 ksi
Buckling *Right Hand Side*	0.0	0.0
Deflection *Right Hand Side*	0.25 in	0.0

Fig. 8.3 is a contour plot showing the design space for this problem. As a first step, we drive to the nominal optimum to the problem, which occurs at the intersection of the boundaries for stress and deflection, shown as a solid circle in the figure.

We then calculate the transmitted variation, given by Eq (8.1), using derivatives that are already available from the nominal optimization. If we calculate the worst-case variation for each constraint using (8.3), and subtract this value from the constraint right hand sides, as in (8.5), the new constraint boundaries are shown as 1*, 2*, and 3* in the figure. The decrease in the feasible region caused by including variation is shaded.

The final step is to drive to the robust optimum, given by the shaded circle in the figure. The optimal value of the objective has increased from 15.8 to 18.0 pounds.

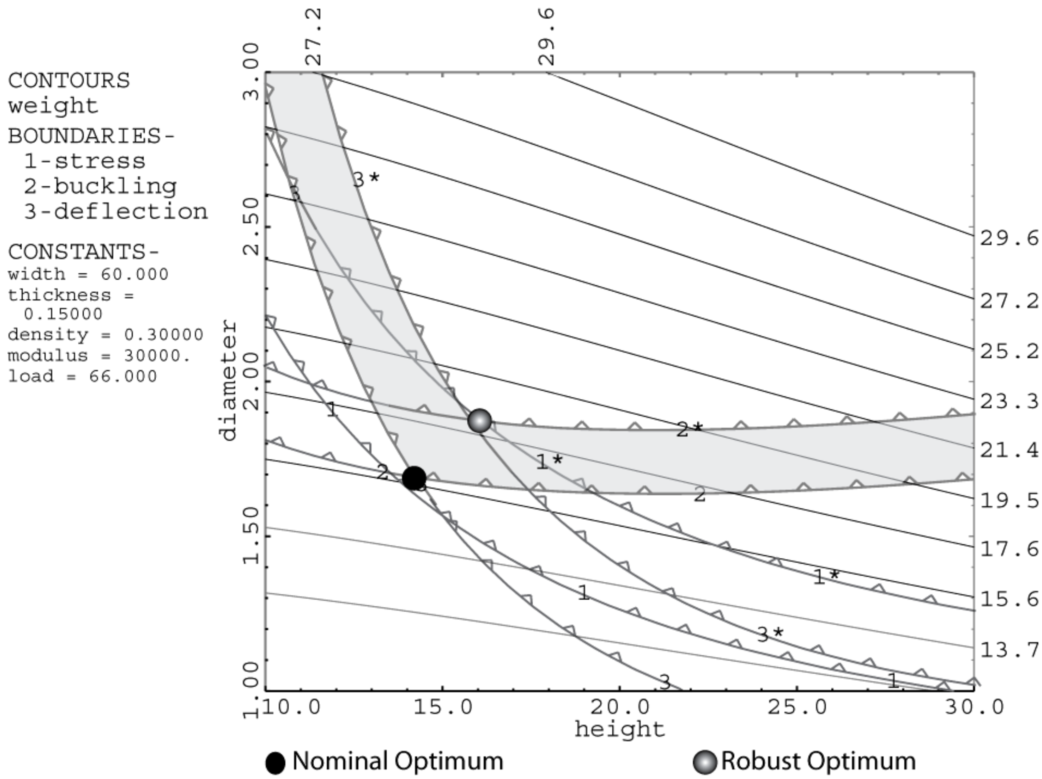


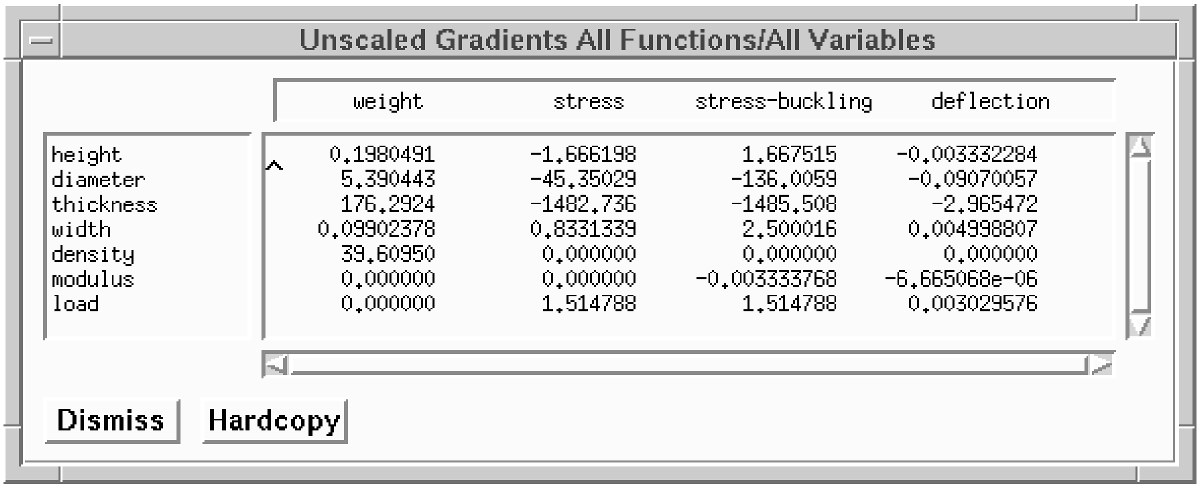
Fig. 8.3 Decrease in feasible region caused by including worst-case tolerances.

2.5.4 A Numerical Example: The Two-barTruss

To illustrate how this method might be implemented "by hand," we will determine the effect on the optimum of the Two-bar Truss problem caused by just two tolerances. For the nominal optimization we have design variables height, diameter, and thickness. We wish to see the effect on the optimum of adding tolerances on the load and the width,

Δ_{load} = 2 kips
Δ_{width} = 1 inch

The first step is to drive to the nominal optimum. The optimum occurs with height = 30, diameter = 2.204, thickness = 0.067, and an optimal weight of 11.88 lbs, with stress and buckling as binding constraints. Next we obtain the derivatives at the optimum using the Gradients window (note that we evaluate un-scaled gradients of all functions with respect to all variables):


Unscaled Gradients All Functions/All Variables

	weight	stress	stress-buckling	deflection
height	0.1980491	-1.666198	1.667515	-0.003332284
diameter	5.390443	-45.35029	-136.0059	-0.09070057
thickness	176.2924	-1482.736	-1485.508	-2.965472
width	0.09902378	0.8331339	2.500016	0.004998807
density	39.60950	0.000000	0.000000	0.000000
modulus	0.000000	0.000000	-0.003333768	-6.665068e-06
load	0.000000	1.514788	1.514788	0.003029576

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The next step is to calculate the transmitted variation to the stress, buckling, and deflection constraints.

$$\Delta_{stress} = \left|(0.833)\cdot(1)\right| + \left|(1.514)\cdot(2)\right| = 3.861$$
$$\Delta_{buckling} = \left|(2.500)\cdot(1)\right| + \left|(1.514)\cdot(2)\right| = 5.528$$
$$\Delta_{deflection} = \left|(0.005)\cdot(1)\right| + \left|(0.00303)\cdot(2)\right| = 0.01106$$

The third step requires that the constraint right hand sides be adjusted. For this problem,

$$\text{stress} \le 100 - 3.861 = 96.139 \text{ ksi}$$
$$\text{buckling} \le 0 - 5.528 = -5.528$$
$$\text{deflection} \le 0.25 - 0.01106 = 0.23894 \text{ in}$$

When we re-optimize in accordance with these new constraints, the new optimum is, height = 29.95, diameter = 2.22, thickness = 0.070, with a weight of 12.36 pounds, and with stress and buckling, again, as binding constraints.

2.6 Verifying the Robust Design: Monte Carlo Simulation

We have discussed a method to develop a robust design. How can we tell if the design is really robust, i.e., how can we be sure that any design within the tolerance bounds will remain feasible? One approach is Monte Carlo simulation, which refers to using a computer to simulate variation in a design. The computer introduces variation within the bounds of the tolerances for each variable. It then calculates the functions. We do this many times--in effect, we have the computer build thousands of designs--and we keep track of how many designs violate the constraints. For a worst-case analysis, the number of infeasible designs should be zero.

Since these are worst-case tolerances, we will assign load and width to have *uniform distributions*. For a uniform distribution, the ends of the range have the same probability of occurring as the middle. So, for example, the load would be assumed to be uniformly distributed with a lower bound of 64 and an upper bound of 68.

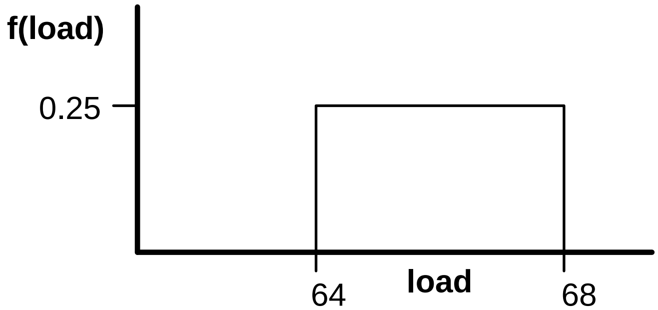


Fig. 8.5 Uniform distribution for load.

The width has a similar distribution, only the lower bound is 59 and the upper bound is 61. We then have the computer generate many designs where load and width are uniformly distributed between their tolerances, and we count the infeasible designs.

The output from running a Monte Carlo simulation (100,000 trials) on the robust design is shown below,

```
no. of trials= 100000
 mean values, variables given
  29.952      60.000      2.2203      .69680E-01  .30000
  30000.      66.000
 mean values, variables calc
  29.951      60.002      2.2203      .69680E-01  .30000
  30000.      66.002
 standard deviations, variables given
  .00000E+00  .57735      .00000E+00  .00000E+00  .00000E+00
  .00000E+00  1.1547
 standard deviations, variables calc
  .49268E-04  .57735      .37418E-05  .15760E-06  .63290E-06
  .00000E+00  1.1517
 mean values, functions
  96.103     -5.5265      .19223
 std devs, functions
  1.7410      2.2150      .43622E-02

 infeasible designs for function 1 =  0
 infeasible designs for function 2 =  0
 infeasible designs for function 3 =  0

 total number of infeasible designs =  0
```

Out of 100,000 simulations, there are no infeasible designs.

It is instructive to compare these results to the non-robust design. If we run the same simulation, with the same tolerances, for the nominal optimum, we get,

```
 no. of trials= 100000
 mean values, variables given
  30.000      60.000      2.2044      .67404E-01  .30000
  30000.      66.000
 mean values, variables calc
  30.000      60.002      2.2044      .67404E-01  .30000
```

```
 30000.      66.002
standard deviations, variables given
 .00000E+00  .57735      .00000E+00  .00000E+00  .00000E+00
 .00000E+00  1.1547
standard deviations, variables calc
 .00000E+00  .57735      .43317E-05  .13389E-06  .63290E-06
 .00000E+00  1.1517
mean values, functions
 99.982     -.32641E-01  .19999
std devs, functions
 1.8111      2.2673      .45353E-02

infeasible designs for function 1 =  49812
infeasible designs for function 2 =  49213
infeasible designs for function 3 =  0

total number of infeasible designs =  56366
```

Out of 100,000 simulations, 56,366 designs had at least one infeasible constraint.

3 Statistical Tolerances

3.1 Introduction

Worst-case analysis is almost always overly conservative. There are some conditions, such as thermal expansion, which must be treated as worst-case. Often, however, it is reasonable to assume that fluctuations are independent random variables. When this is the case, it is very unlikely they will simultaneously occur in the worst possible combination. With a statistical tolerance analysis, the low probability of a worst-case combination can be taken into account. By allowing a small number of *rejects*--infeasible designs--the designer can use larger tolerances, or, as will be shown, back away from the optimum design a smaller amount than for a worst-case analysis.

3.2 Background

For this situation, variables with tolerances will be treated as random variables. Typically a random variable is described by a distribution type and distribution characteristics such as the mean and variance (or standard deviation, which is the square root of variance). We will consider all of the variables which have tolerances to be random variables described by normal distributions, with a mean at the nominal value and specified standard deviation.

3.3 Calculating Transmitted Variation: Linear Statistical Analysis

Just as with worst-case tolerances, we will transmit the variation of the variables into the constraints. Once again, we will rely on a first-order Taylor series, only this time we will find the mean and variance of the series. The mean of a first-order Taylor series is just the nominal value; the variance is given by:

$$\sigma_{g_i}^2 = \sum_{j=1}^{n} \left(\frac{\partial g_i}{\partial x_j} \sigma_{x_j} \right)^2 + \sum_{j=1}^{l} \left(\frac{\partial g_i}{\partial p_j} \sigma_{p_j} \right)^2 \qquad (8.7)$$

$$\sigma_i^2 = \sigma_{b_i}^2 + \sigma_{g_i}^2 \qquad (8.8)$$

We will refer to σ_i^2 as the "total constraint variance."

Thus we will consider that randomness in the variables *induces* randomness in the functions. The constraint boundaries are no longer described by a sharp boundary but rather by a distribution, with the mean at the nominal value, and with one tail inside the feasible region and one tail outside the feasible region. This is illustrated in Fig. 8.6a.

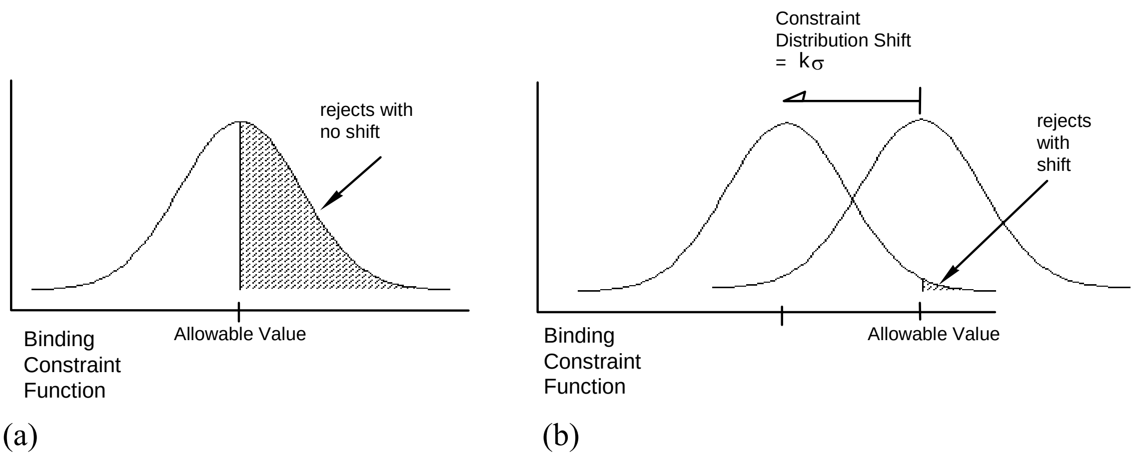


Fig. 8.6 The distribution of a constraint function. (a) The distribution with no shift. (b) The distribution with a shift to reduce the number of rejects.

We note that the distributions shown in Fig. 8.6 are normal. This is an assumption of this method. An important theorem in statistics, the Central Limit Theorem, states that sums and differences of random variables, regardless of their distributions, will tend to be normal. With engineering models, variables combine not only as sums and differences but as products, quotients, powers, etc. This means the assumption that the functions are normally distributed will only be approximately satisfied.

3.4 Developing a Robust Optimal Design with Statistical Tolerances

3.4.1 Compensating for Statistical Variation

As with worst-case tolerances, we will modify the constraints to take into account the transmitted variation. Thus each constraint becomes,

$$g_i + k\sigma_i \le b_i \tag{8.9}$$

Alternatively, we can consider that the variation has reduced the right hand side,

$$g_i \le b_i - k\sigma_i \tag{8.10}$$

where k is the number of standard deviations we decide to shift. Unlike the worst-case approach where we wanted to be feasible 100% of the time, with the statistical approach we set the level of feasibility we desire. For a normal distribution,

Table 2. Relation of *k* to Constraint Feasibility	
Value of *k* (number of standard deviations)	Percentage of Designs that will be Feasible (Normal Distribution)
1	84.13
2	97.725
3	99.865
4	99.9968

Shifting a constraint to control the number of rejects is illustrated in the Fig. 8.6b.

To obtain a robust optimum based on a statistical analysis, we follow essentially the same steps as with the worst-case approach,

1. Drive to the nominal optimum.
2. Calculate the transmitted variance.
3. Reduce the allowable value for a constraint by the amount of *k* times the standard deviation for a *less than* constraint; increase it for a *greater than* constraint. This has the effect of shifting the constraint distribution into the feasible region, as shown in Fig. 8.6b.
4. Assuming the transmitted variation is a constant, re-optimize to find the robust optimum.
5. Estimate the overall estimated feasibility as the product of the feasibilities for the binding constraints.

The assumptions of this approach include,

- Variables are independent and normally distributed.
- The robust optimum is close to the nominal optimum.
- Derivatives are constant, i.e., second derivatives are equal to zero. This assumption is consistent with assuming a linear Taylor expansion, (8.7), for the transmitted variation.
- Constraints are normally distributed.
- Constraints are not correlated. This means that for a random perturbation, the probability of one constraint being violated is independent of other constraints. This allows us to multiply the probability of each constraint together (step 5 above) to get the overall feasibility.

These assumptions are not always completely met, so we consider this method as a means of *estimating the order of magnitude of the number of rejects*. This means we will determine whether the per cent rejects will be 10%, 1%, 0.1%, etc. This level of accuracy is usually on a par with the accuracy of tolerance data available during the design stage.

3.4.2 An Example: The Two-bar Truss

In this section we will apply the method to the Two-bar truss. The tolerances are given in Table 3 below. For comparison to a worst-case tolerance analysis, the standard deviations are made to be one third the worst-case tolerances given in the previous section. This means a worst-case tolerance band would be $\pm 3\sigma$.

Table 3 Statistical Tolerance Data for the Two-bar Truss

Description	Nominal Value	Standard Deviation
Height, H	Design Variable	1.67 in
Width, B	60 in	0.167 in
Diameter, d	Design Variable	0.033 in
Thickness, t	0.15 in	0.0033 in
Modulus	30000 ksi	500 ksi
Density	0.3 lb/in^3	0.0033 lb/in^3
Load, P	66 kips	1 kips
Yield Strength *Right Hand Side*	100 ksi	1.67 ksi
Buckling *Right Hand Side*	0.0	0.0
Deflection *Right Hand Side*	0.25 in	0.0

We desire each constraint to be feasible 99.865% of the time. To effect this, we calculate the variance for each constraint using (8.7). In the case of stress, which has a tolerance on the right hand side, we add in that variance using (8.9). We then subtract 3σ from the constraint right hand sides, as shown in Eq (8.10).

Fig. 8.7 is a contour plot showing the design space for this problem. The shaded area shows the decrease in the feasible region caused by the tolerances. The new constraint boundaries are shown as 1*, 2* and 3*. Comparing to Fig. 8.3, we see that the decrease is smaller than for worst-case tolerances. The optimal value of the objective has increased from 15.8 to 16.8 pounds.

We have two binding constraints that should each be feasible 99.865% of the time. The predicted overall feasibility is computed to be $0.99865 \times 0.99865 = 0.9973$ or 99.73%. Monte Carlo simulation of the robust optimum gives a feasibility of 99.8%.

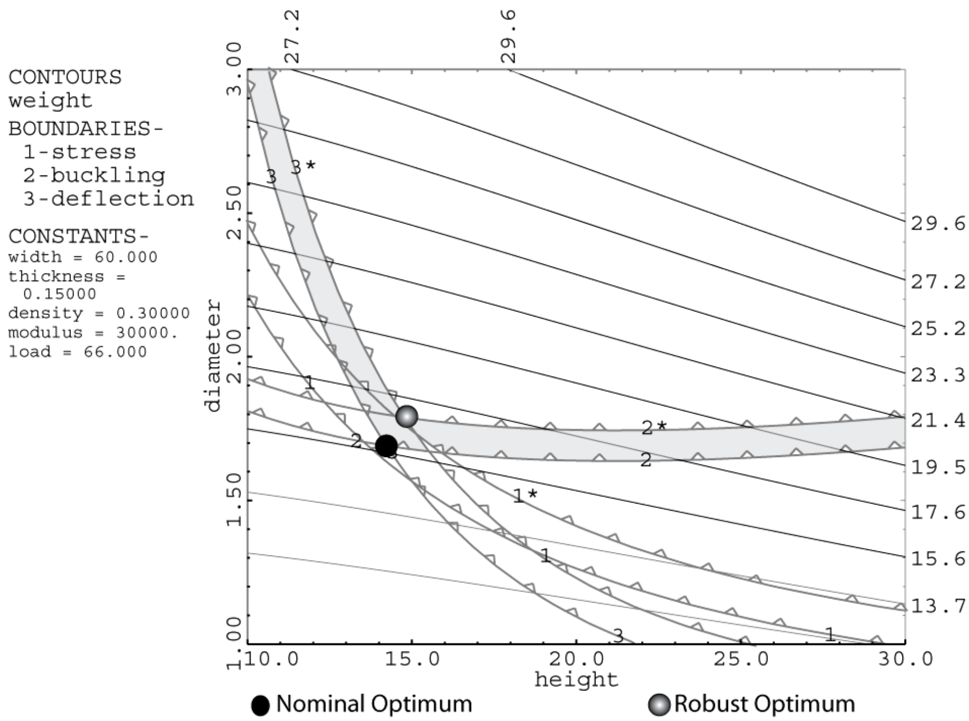


Fig. 8.7 Effect of statistical tolerances on optimum for Two-bar Truss problem. Shaded area is decrease in feasible region caused by including statistical tolerances. Compare to Fig. 8.3.

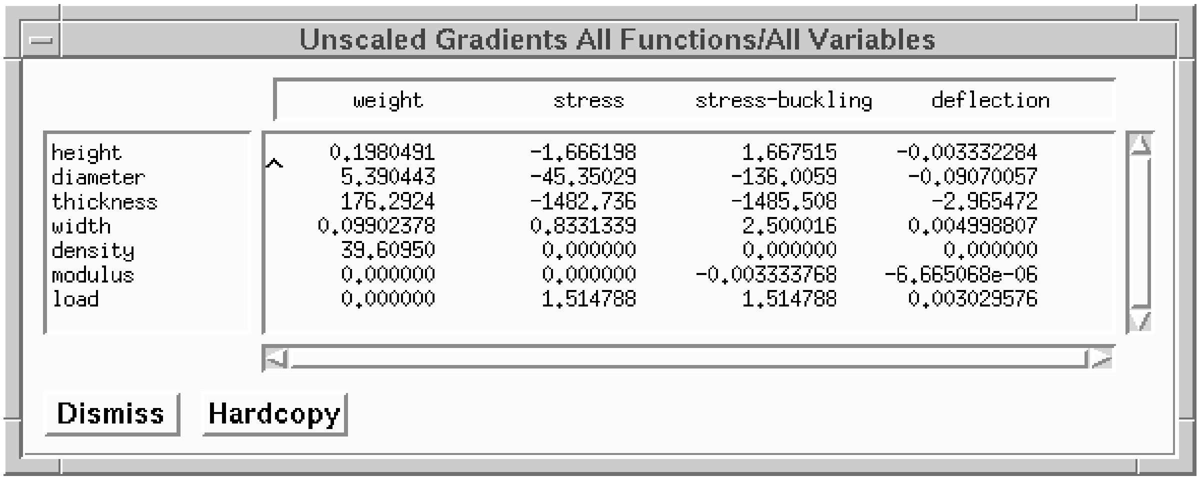
3.4.3 <u>A Numerical Example: The Two-barTruss</u>

As we did for worst-case tolerances, we will show how to implement this method by hand. For this problem we have as design variables height, diameter, and thickness. The optimum occurs with height = 30, diameter = 2.2044, thickness = 0.06740, and an optimal weight of 11.88 lbs, with stress and buckling as binding constraints.

We wish to see the effect on the optimum of adding tolerances on the load and the width,

$$\sigma_{load} = 0.6667 \ kips$$
$$\sigma_{width} = 0.3333 \ inch$$

The first step is to drive to the nominal optimum, which we have already done.
The second step is to calculate the transmitted variation using (8.7) with derivatives at the optimum:


Unscaled Gradients All Functions/All Variables

	weight	stress	stress-buckling	deflection
height	0.1980491	-1.666198	1.667515	-0.003332284
diameter	5.390443	-45.35029	-136.0059	-0.09070057
thickness	176.2924	-1482.736	-1485.508	-2.965472
width	0.09902378	0.8331339	2.500016	0.004998807
density	39.60950	0.000000	0.000000	0.000000
modulus	0.000000	0.000000	-0.003333768	-6.665068e-06
load	0.000000	1.514788	1.514788	0.003029576

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We estimate the variance of the functions as:

$$\sigma^2_{stress} = (0.833 \times 0.333)^2 + (1.514 \times 0.667)^2 = 1.094$$

$$\sigma^2_{buckling} = (2.500 \times 0.333)^2 + (1.514 \times 0.667)^2 = 1.713$$

$$\sigma^2_{deflection} = (0.005 \times 0.333)^2 + (0.00303 \times 0.667)^2 = 0.000006857$$

We will shift the constraints by 3σ. These amounts are,

$$3\sigma_{stress} = 3.138$$

$$3\sigma_{buckling} = 3.926$$

$$3\sigma_{deflection} = 0.007856$$

We subtract these amounts from the right hand sides, as in (8.10),

$$\text{stress} \le 100 - 3.138 = 96.862 \text{ ksi}$$

$$\text{buckling} \le 0 - 3.926 = -3.926$$

$$\text{deflection} \le 0.25 - 0.007856 = 0.24214 \text{ in}$$

When we re-optimize the new optimum is, height = 29.98, diameter = 2.21, thickness = 0.0694, with a weight of 12.27 pounds, and with stress and buckling as binding constraints.

3.5 Verifying the Robust Design with Monte Carlo Simulation

We verify these values using a Monte Carlo simulation similar to the one describe for worst-case tolerances, only the independent variables are given normal distributions instead of uniform distributions.

The output from running this program is given below,

```
no. of trials= 100000
 mean values, variables given
```

```
 29.980      60.000      2.2122      .69350E-01  .30000
 30000.      66.000
mean values, variables calc
 29.980      60.000      2.2122      .69350E-01  .30000
 30000.      65.997
standard deviations, variables given
 .00000E+00  .33330      .00000E+00  .00000E+00  .00000E+00
 .00000E+00  .66670
standard deviations, variables calc
 .38388E-04  .33262      .52221E-05  .47925E-07  .63290E-06
 .00000E+00  .66709
mean values, functions
 96.860     -3.9271      .19373
std devs, functions
 1.0160      1.2838      .25398E-02

infeasible designs for function 1 =  118
infeasible designs for function 2 =  118
infeasible designs for function 3 =  0

total number of infeasible designs =  183
```

We have 0.183% infeasible designs. We predicted,

$$1-(0.99865\times 0.99865)=0.00270=0.27\%$$

This is well within our desired order of magnitude accuracy.

4 Minimizing Variance: Sensitivity Robustness

4.1 Introduction

Up to this point we have considered only *feasibility robustness*: we wanted to develop designs that could tolerate variation and still work. We developed a method based on a linear approximation of transmitted variation.

For worst-case analysis, we estimate transmitted variation by,

$$\Delta g_i=\sum_{j=1}^{n}\left|\frac{\partial g_i}{\partial x_j}\Delta x_j\right|+\sum_{j=1}^{m}\left|\frac{\partial g_i}{\partial p_j}\Delta p_j\right|$$

For a statistical analysis, we estimate transmitted variation by,

$$\sigma_{g_i}^2=\sum_{j=1}^{n}\left(\frac{\partial g_i}{\partial x_j}\sigma_{x_j}\right)^2+\sum_{j=1}^{l}\left(\frac{\partial g_i}{\partial p_j}\sigma_{p_j}\right)^2$$

Besides feasibility robustness, we might also be interested in *sensitivity robustness,* which refers to *reducing the sensitivity of the design to variation*. This can be achieved by minimizing the transmitted variation as an objective in our optimization problem, either as the sole objective in the problem or in combination with other objectives related to performance. The transmitted variation might also be a constraint.

The idea here is to find a region in the design space where *the derivatives of the function are small.* Thus we can reduce the effect of the tolerances *without reducing the tolerances themselves*. This idea is very similar to the central concept of Taguchi methods. Taguchi showed that it was possible to reduce variation in a product without reducing tolerances (which usually costs money) by moving to a different spot in the design space, where the design was less sensitive to the tolerances. In contrast to the computer models we are using, Taguchi based his method on using Design of Experiments to obtain models experimentally.

Minimizing variance can be computationally expensive, since we are minimizing a function which is composed of derivatives. To obtain a search direction, we will need to take second derivatives.

4.2 Example of Minimizing Variation

We will illustrate the concept of reducing transmitted variation by considering the design of a check valve—a device made to restrict flow to one direction only.

A diagram of a check valve is shown in Fig. 8.8. The purpose of the valve is to allow fluid flow in only one direction. Fluid can flow through the valve from left to right when the pressure of the flow overcomes the force exerted on the ball by the spring. The pressure required to unseat the ball is called the “cracking pressure.” It is desirable to minimize cracking pressure to reduce pressure drop across the valve; however, cracking pressure must be sufficient to prevent backflow. Design variables, parameters and tolerances for this problem are given in Table 4.

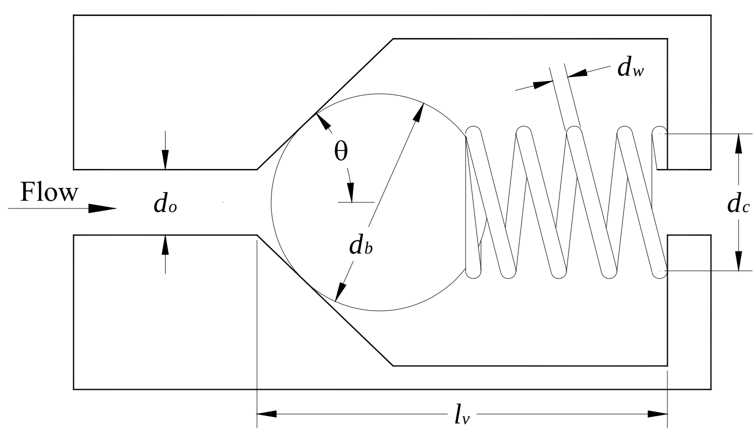

Fig. 8.8 Diagram of a check valve.

The design problem is to choose values for the variables to improve the sensitivity robustness of cracking pressure (i.e. reduce its sensitivity to variation), subject to constraints on cracking pressure, stress at full deflection, ball travel (spring compression) and various diameter ratios.

Table 4. Variables, Parameters and Tolerances for Check Valve

Design Variables	**Tolerance**	**Standard Deviation**
Seat angle, θ (degrees)	±1.0	0.3333
Ball diameter , d_b (cm)	± 0.025	0.008333
Spring coil diameter, d_c (cm)	± 0.05	0.016667
Spring wire diameter, d_w (cm)	± 0.005	0.0016667
Spring length unloaded (cm)	± 0.1	0.03333
Spring number of coils	±0.5	0.16667
Parameters		
Length of valve, l_v (cm)	±0.1	0.03333
Diameter of orifice, d_v (cm)	± 0.025	0.008333
Shear modulus (kPa/m^2)	± 7.e5	2.333e5

We will need to combine sensitivity robustness with feasibility robustness to insure that cracking pressure is at least 15 kPa (about 2.2 psi) for 99% of all valves. This can be accomplished in two steps, where we first minimize variance and then add constraint shifts and re-optimize to achieve feasibility robustness. The starting design is given in the first column of Table 5.

After determining the minimum variance design, the next step was to shift it to obtain feasibility robustness. Constraint shifts based on a linear estimate of variance were inadequate for this problem so variance was calculated using a second order model. The shifted design is given in column 2 of Table 5.

Also shown in Table 5 is a comparison design. In order to determine the effect of minimizing variance we wanted to compare it to some sort of baseline design. We chose as a comparison design the design that results from maximizing ball travel as the objective, ignoring variance, and with all other constraints the same. The comparison design was then shifted to obtain feasibility robustness.

Table 5. Starting, Minimum Variance and Comparison Designs

Design Variables	**Starting Design**	**Shifted Min Variance Design**	**Comparison Design***
Seat angle, θ (degrees)	45.	34.9	43.5
Ball diameter , d_b (cm)	1.25	1.47	0.720
Spring coil diameter, d_c (cm)	1.0	1.13	0.535
Spring wire diameter, d_w (cm)	0.075	0.0755	0.0379
Spring length unloaded (cm)	3.0	2.00	2.77
Spring number of coils	10	8	12
Parameters			
Length of valve, l_v (cm)	2.5	2.5	2.5
Diameter of orifice, d_v (cm)	0.635	0.635	0.635
Shear modulus (kPa/m^2)	8.3e7	8.3e7	8.3e7
Objective			

Variance of cracking pressure (kPa)	8.70	2.46	5.63
Constraints			
Cracking pressure ≥ 15 (kPa)	77	20.02 (b)	25.37 (b)
0.5 ≤ Ball diam/coil diam ≤ 0.8	0.8 (b)	0.77 (b)	0.74 (b)
4 ≤ Coil diam/wire diam ≤ 16	13.3	15.03 (b)	14.1 (b)
Ball travel ≥ 0.5 (cm)	0.8	0.60 (b)	1.63
Stress ≤ 900000 (kPa)	494000	295000	746600 (b)
Predicted Feasibility	N/A	96.06%	96.06%
Actual Feasibility	N/A	96.42%	96.59%

The symbol "(b)" indicates a binding constraint
*the objective for the comparison design was to maximize ball travel.

Monte Carlo simulation was used to verify robustness, and the results are shown in the last two rows of the table. During the simulation, pressure values were recorded so that the distributions could be graphed. These are shown in Fig. 8.9. The improvement of the minimum variance design over the comparison design is clearly evident.

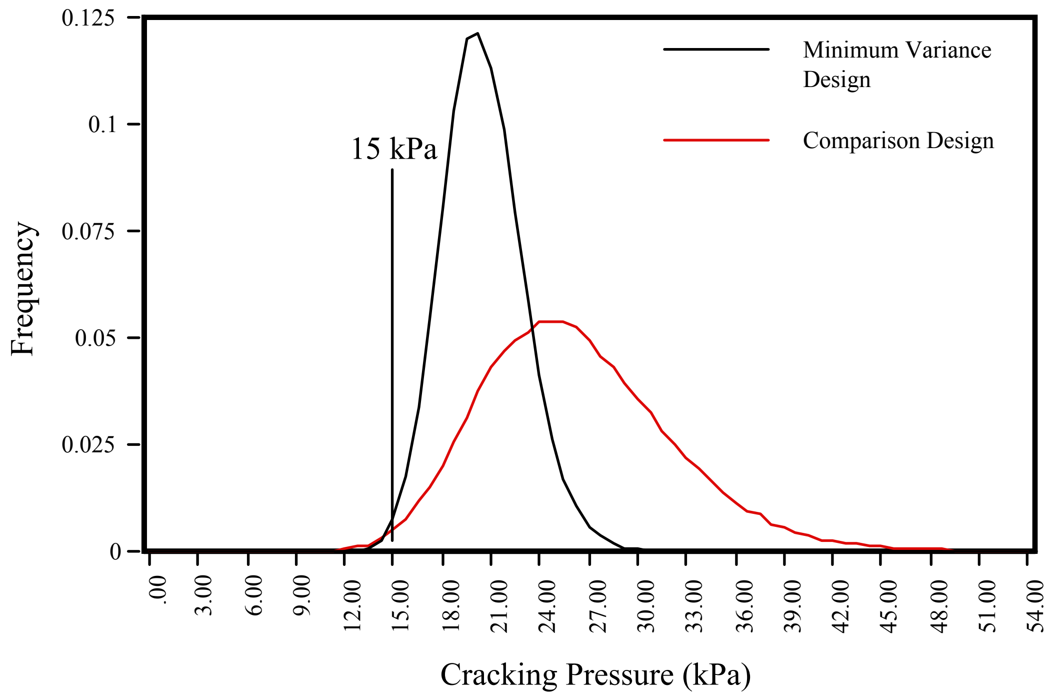

Fig. 8.9 Minimum variance and comparison design distributions for cracking pressure

5 References

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