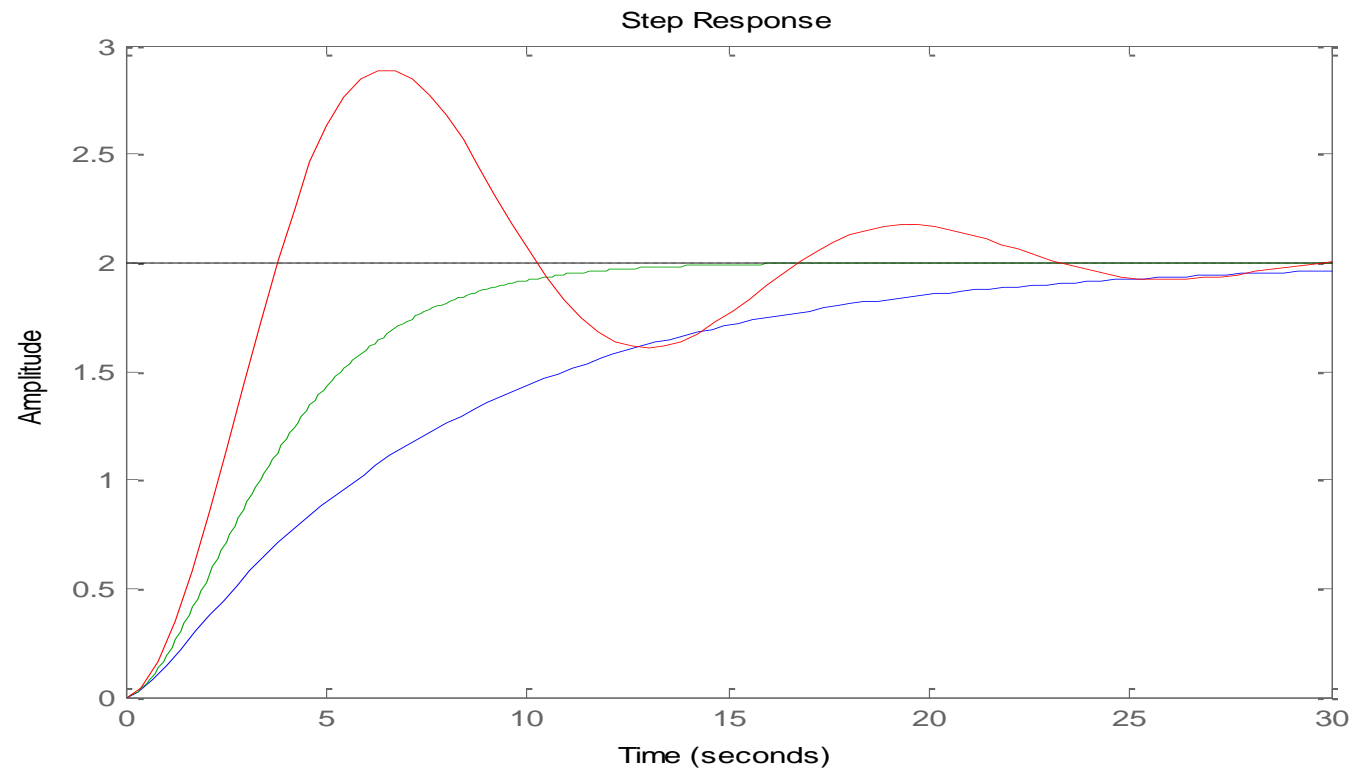


Second Order Systems



Second Order Equations

Standard Form

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

K = Gain

τ = Natural Period of Oscillation

ζ = Damping Factor (zeta)

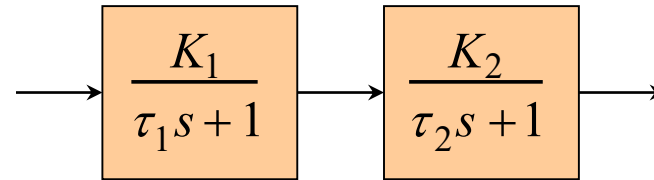
Note: this has to be 1.0!!!

Corresponding Differential Equation

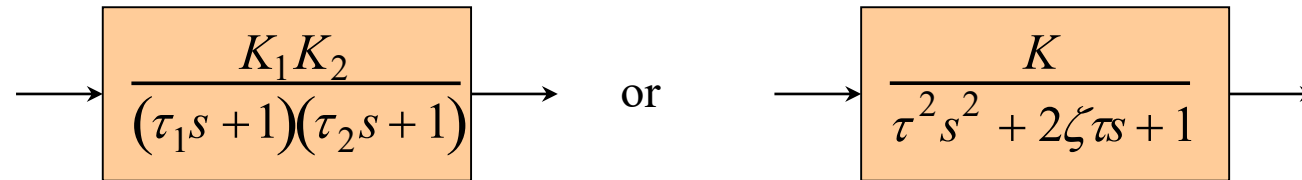
$$\tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = Ku(t)$$

Origins of Second Order Equations

1. Multiple Capacity Systems in Series



become



2. Controlled Systems (to be discussed later)

3. Inherently Second Order Systems

- Mechanical systems and some sensors
- Not that common in chemical process control

Examination of the Characteristic Equation

$$\tau^2 s^2 + 2\zeta\tau s + 1 = 0$$

$\zeta > 1$	Overdamped	Two distinct real roots
$\zeta = 1$	Critically Damped	Two equal real roots
$0 < \zeta < 1$	Underdamped	Two complex conjugate roots

Response of 2nd Order System to **Step** Inputs

Overdamped Eq. 5-48 or 5-49	Sluggish, no oscillations
Critically damped Eq. 5-50	Faster than overdamped, no oscillation
Underdamped Eq. 5-51	Fast, oscillations occur

Ways to describe underdamped responses:

- Rise time
- Settling time
- Decay ratio
- Time to first peak
- Overshoot
- Period of oscillation

Response of 2nd Order Systems to **Step** Input ($0 < \zeta < 1$)

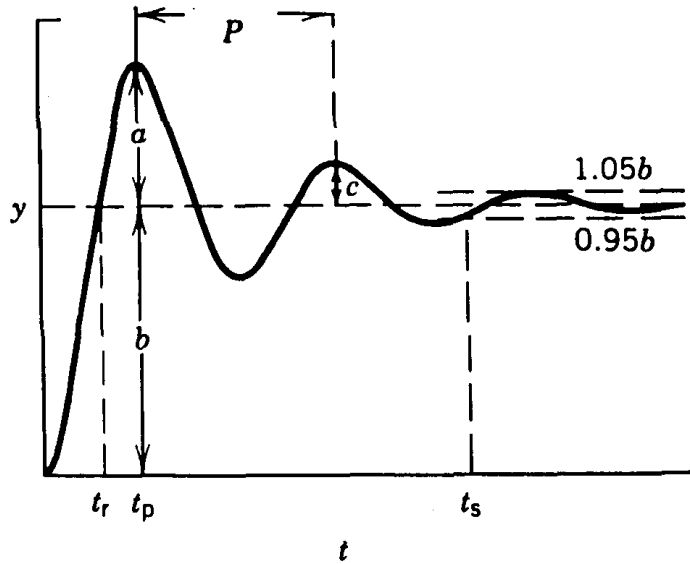


Figure 5.10. Performance characteristics for the step response of an underdamped process.

1. **Rise Time:** t_r is the time the process output takes to first reach the new steady-state value.
2. **Time to First Peak:** t_p is the time required for the output to reach its first maximum value.
3. **Settling Time:** t_s is defined as the time required for the process output to reach and remain inside a band whose width is equal to $\pm 5\%$ of the total change in y . The term 95% response time sometimes is used to refer to this case. Also, values of $\pm 1\%$ sometimes are used.
4. **Overshoot:** $OS = a/b$ (% overshoot is $100a/b$).
5. **Decay Ratio:** $DR = c/a$ (where c is the height of the second peak).
6. **Period of Oscillation:** P is the time between two successive peaks or two successive valleys of the response.

Eq. 5-51

$$y(t) = KM \left\{ 1 - e^{-\zeta t/\tau} \left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau} t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau} t\right) \right] \right\}$$

Response of 2nd Order Systems to **Step** Input

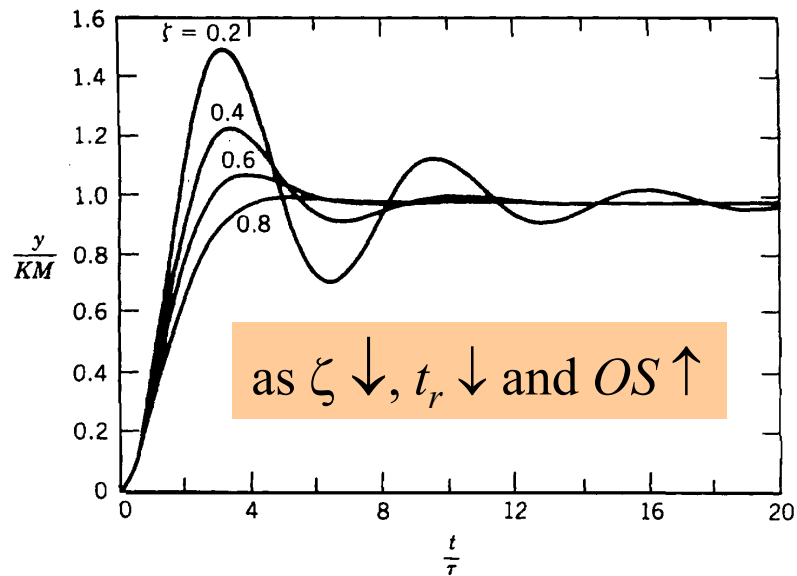


Figure 5.8. Step response of underdamped second-order processes.

$$0 < \zeta < 1$$

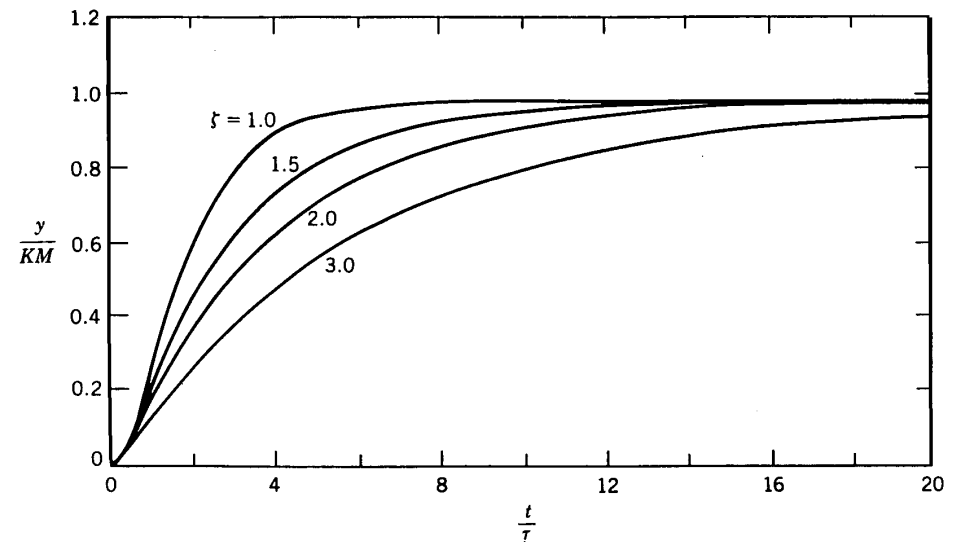


Figure 5.9. Step response of critically-damped and overdamped second-order processes.

$$\zeta \geq 1$$

Note that $\zeta < 0$ gives an unstable solution

Relationships between OS, DR, P and τ , ζ

for **step** input to 2nd order system,
underdamped solution

$$Y(s) = \frac{KM}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}, \quad \zeta < 1$$

(5-52)	$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$		
(5-53)	$OS = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$	$\zeta = \sqrt{\frac{[\ln(OS)]^2}{\pi^2 + [\ln(OS)]^2}}$	Above (5-56)
(5-54)	$DR = (OS)^2$ $= \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$		
(5-55)	$P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$	$\tau = \frac{\sqrt{1-\zeta^2}}{2\pi} P$	Above (5-57)
(5-60)	$t_r = \frac{\tau}{\sqrt{1-\zeta^2}} (\pi - \cos^{-1} \zeta)$		

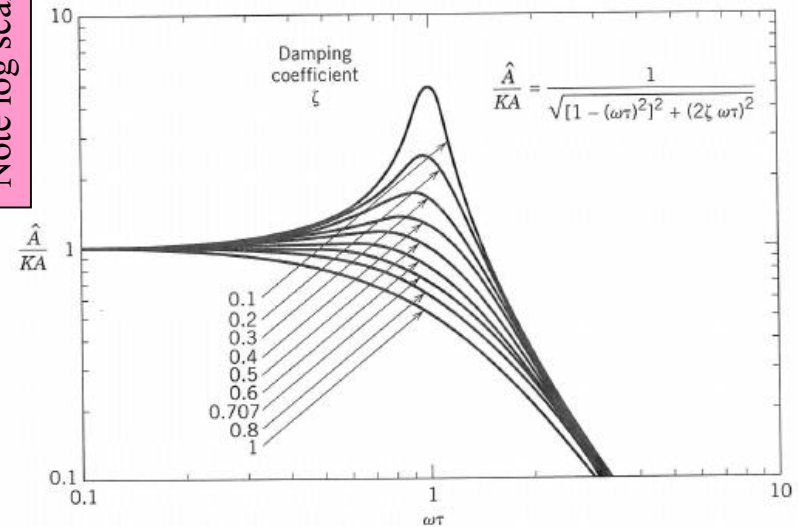
Response of 2nd Order System to Sinusoidal Input

- ✱ Output is also oscillatory
- ✱ Output has a different amplitude than the input
- ✱ Amplitude ratio is a function of ζ , τ
(see Eq. 5-63)
- ✱ Output is phase shifted from the input
- ✱ Frequency ω must be in radians/time!!!
(2π radians = 1 cycle)
- ✱ $P = \text{time/cycle} = 1/(\nu)$, $2\pi\nu = \omega$, so $P = 2\pi/\omega$
(where ν = frequency in cycles/time)

Sinusoidal Input, 2nd Order System (Section 5.4.2)

- Input = $A \sin \omega t$, so
 - A is the amplitude of the input function
 - ω is the frequency in radians/time
- At long times (so exponential dies out),

Note log scale



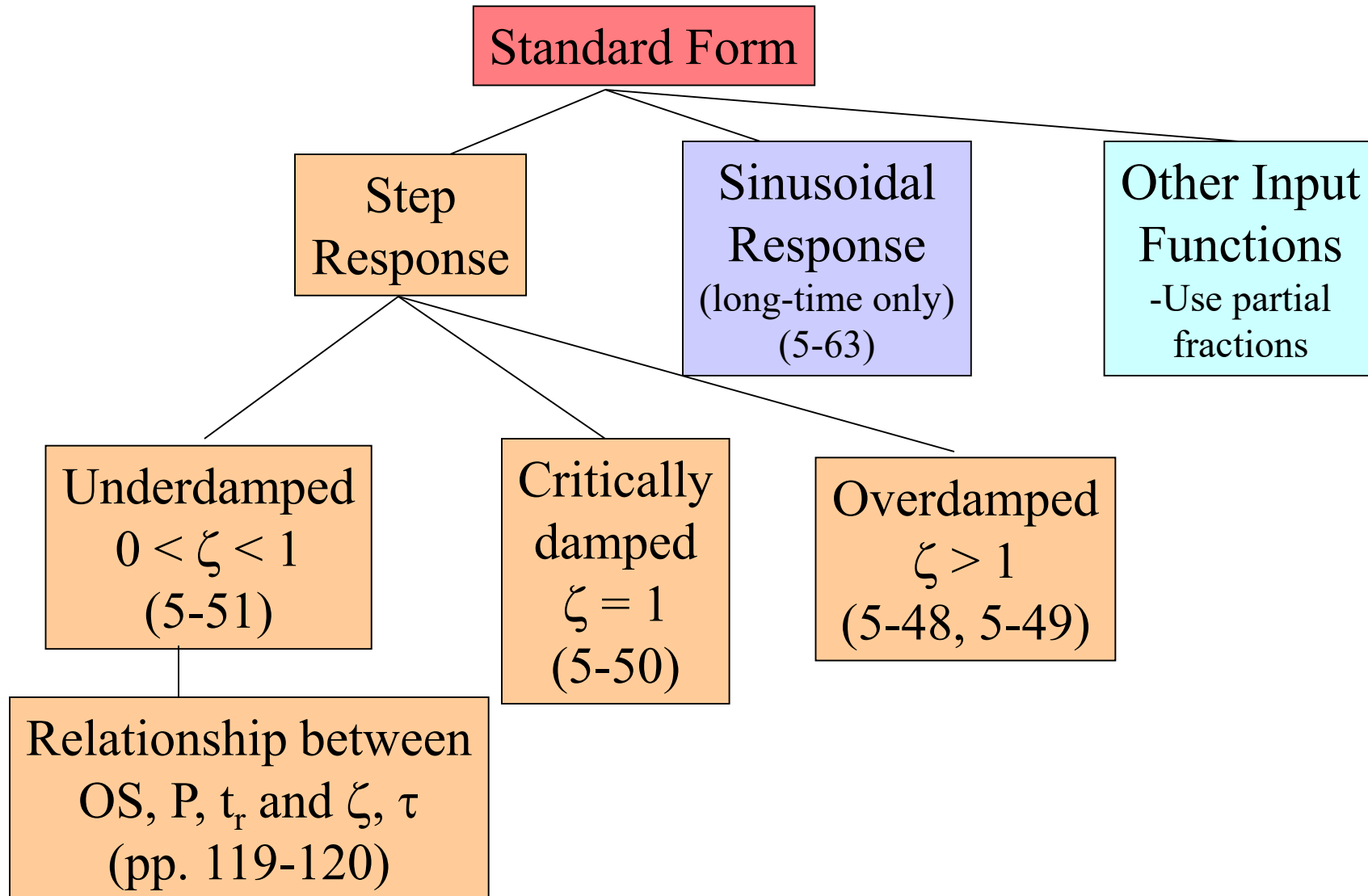
\hat{A} is the output amplitude

$$\hat{A} = \frac{KA}{\sqrt{[1 - (\omega\tau)^2]^2 + (2\zeta\omega\tau)^2}} \quad (5-63)$$

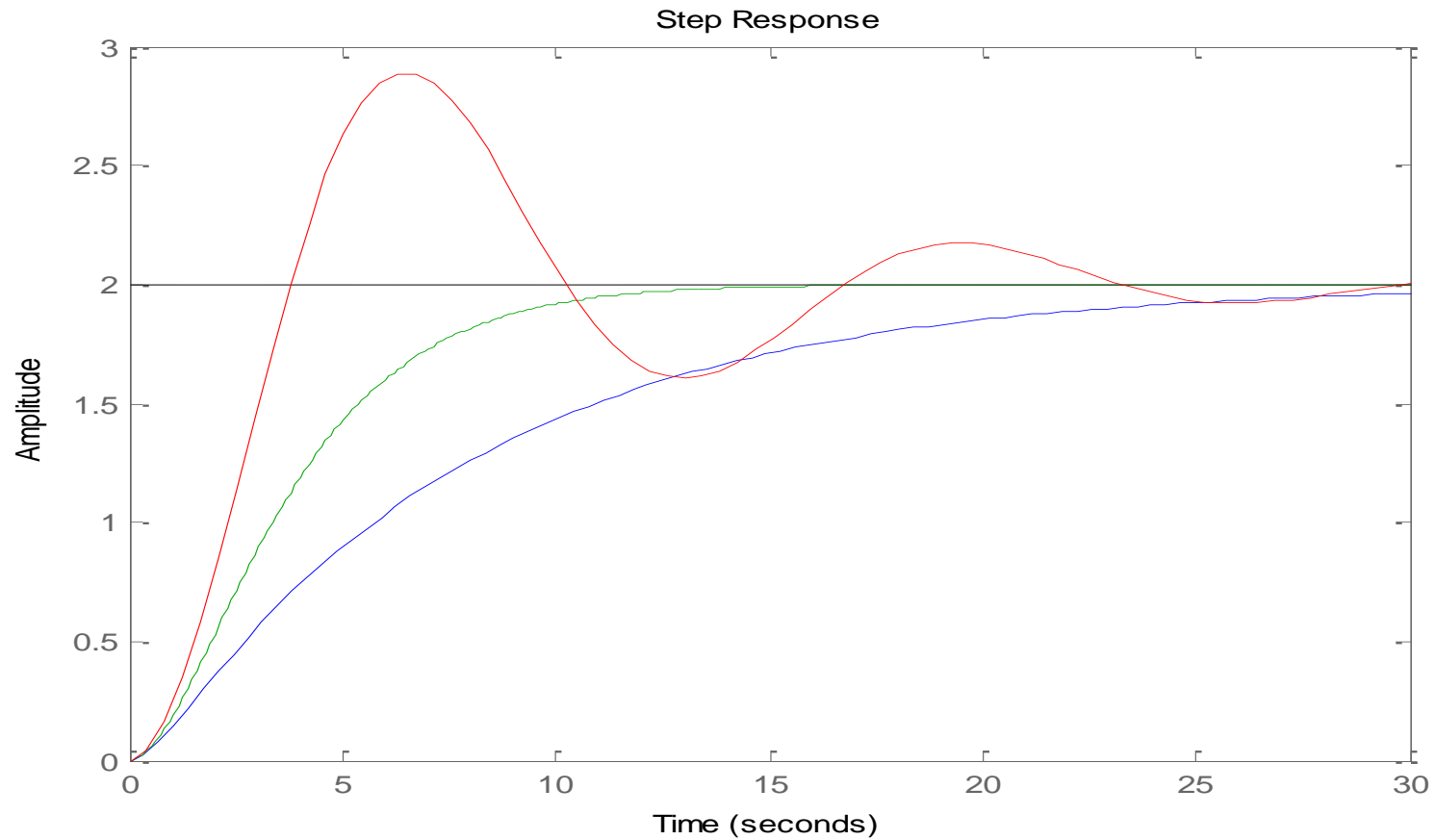
Note: There is also an equation for the maximum amplitude ratio (5-66)

Bottom line: We can calculate how the output amplitude changes due to a sinusoidal input

Road Map for 2nd Order Equations



Determine 2nd Order System



Example 5.5

- Heated tank + controller = 2nd order system

(a) When feed rate changes from 0.4 to 0.5 kg/s (step function), T_{tank} changes from 100 to 102°C.

Find gain (K) of transfer function:

Example 5.5

- Heated tank + controller = 2nd order system
- (b) Response is slightly oscillatory, with first two maxima of 102.5 and 102.0°C at 1000 and 3600 S.
- What is the complete process transfer function?**

Example 5.5

- Heated tank + controller = 2nd order system

(c) Predict t_r :

Example 5.6

- Thermowell + CSTR = 2nd order system

$$(a) \frac{T'_{meas}(s)}{T'_{reactor}(s)} = \frac{1}{(3s + 1)(10s + 1)}$$

Find τ , ζ :

