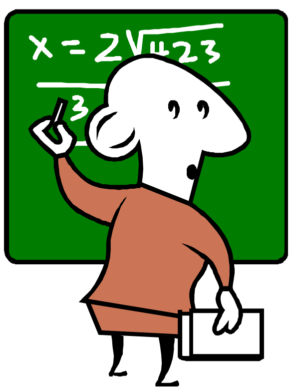
Class 17

More Laplace



Initial Value

$$\frac{(s+2)}{(s+3)(s+4)}$$

Multiply by s and set s = ∞

$$\frac{s(s+2)}{(s+3)(s+4)} = \left[\frac{1\left(1+\frac{2}{s}\right)}{\left(1+\frac{3}{s}\right)\left(1+\frac{4}{s}\right)}\right]_{s\to\infty} = 1$$

Divide both top and bottom by s^2

Final Value

$$\frac{(s+6)}{(s+1)(s+2)}$$

Multiply by s and set s = 0

$$\frac{s(s+6)}{(s+1)(s+2)} = \left[\frac{s(s+6)}{(s+1)(s+2)}\right]_{s\to 0} = 0$$

Complex Factors

- Denominator may have complex roots
  - $s^2 + d_1 s + d_0$ where $d_1^2/4 < d_0$
  - Remember quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Example: $s^2 + 4s + 5$

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$$

$$(s+2+j)(s+2-j)$$

$$or \quad s = -2 - j \;\; and \;\; -2 + j$$

Implications of Complex Factors

- Complex roots indicate oscillatory behavior
- If the sign of the real part of the complex roots is negative, convergence is expected
  - Conversely, if the real part is positive, it will diverge
- Algebra needed to invert transforms with complex roots is messy but doable
- We don’t need to invert the transform to tell whether it will converge or diverge, or whether or not it will oscillate

Practice

- Will y(t) converge or diverge? Is y(t) smooth or oscillatory?

$$Y(s) = \frac{s+2}{s(s^2+4s+13)}$$

Method 1: $s^2 + 4s + 13 = (s+2)^2 + 9 \Rightarrow \text{oscillatory}$

Method 2: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2 \cdot 1} = \dfrac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3j$

oscillatory, converging

Inverting Transforms with Complex Roots in the Denominator

- There are at least two different ways to proceed as described in your text:
  - Expansion without using complex numbers, followed by completing the square to invert the transform (preferred)
    - Example 3.4
  - Use of complex numbers and Euler's identity (p. 43)
    - $\cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2$; $\sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2$

Completing the Square

This is the original equation.	$x^2 + 6x - 7 = 0$
Move the loose number over to the other side.	$x^2 + 6x \quad = 7$
Take half of the x-term (that is, divide it by two) (and don't forget the sign!), and square it. Add this square to both sides of the equation.	$x^2 + 6x + 9 = 7 + 9$ $+3 \rightarrow +9$
Convert the left-hand side to squared form. Simplify the right-hand side.	$(x + 3)^2 = 16$

http://www.purplemath.com/modules/sqrquad.htm

Example 1

$$Y(s) = \frac{s+2}{s(s^2+4s+5)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2+4s+5}$$

- Find α_1: $$\alpha_1 = \left[\frac{s(s+2)}{s(s^2+4s+5)}\right]_{s=0} = \frac{2}{5}$$
- To get α_2 and α_3, clear denominator and match "like" terms

$$s+2 = \alpha_1(s^2+4s+5) + s(\alpha_2 s + \alpha_3) = \underline{(\alpha_1+\alpha_2)}s^2 + \underline{(4\alpha_1+\alpha_3)}s + \alpha_1 5$$

- s^2 terms → $\alpha_1 + \alpha_2 = 0$, so $\alpha_2 = -2/5$
- s terms → $4\alpha_1 + \alpha_3 = 1$, so $\alpha_3 = -3/5$

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{s^2+4s+5}$$

Complete the square
Put into proper form for inversion

- Wanted: $s^2 + 4s + 5 = (s+b)^2 + w^2$

- How?

  b = (coefficient in front of s term)/2 = 4/2 = 2

- Knowing b, find w

  $b^2 + w^2 = 5 = 4 + w^2$, so w = 1

$$Y(s) = \frac{2}{5s} + \frac{-\frac{2}{5}s - \frac{3}{5}}{(s+2)^2 + 1}$$

Example 1 (cont)

Need to Get Form in Laplace Table

$$L\{e^{-bt}\cos(\omega t)\} = \frac{s+b}{(s+b)^2+\omega^2}$$

#15 in Table 3.1

$$L\{e^{-bt}\sin(\omega t)\} = \frac{\omega}{(s+b)^2+\omega^2}$$

#14 in Table 3.1

$$\frac{-\frac{2}{5}s-\frac{3}{5}}{(s+2)^2+1}$$

Has both an s and a number on the top

$$\frac{-\frac{2}{5}s-\frac{3}{5}}{(s+2)^2+1} = \frac{-\frac{2}{5}(s+2)+\frac{1}{5}}{(s+2)^2+1} = -\frac{2}{5}\left[\frac{(s+2)}{(s+2)^2+1}\right] + \frac{1}{5}\left[\frac{1}{(s+2)^2+1}\right]$$

Finally:

$$Y(s) = \frac{2}{5s} - \frac{2}{5}\left[\frac{(s+2)}{(s+2)^2+1}\right] + \frac{1}{5}\left[\frac{1}{(s+2)^2+1}\right]$$

and inverting

$$y(t) = \frac{2}{5} - \frac{2}{5}e^{-2t}\cos t + \frac{1}{5}e^{-2t}\sin t$$

Example 1 (cont)

Analyze the Equation

$$y(t) = \frac{2}{5} - \frac{2}{5} e^{-2t} \cos t + \frac{1}{5} e^{-2t} \sin t$$

- e^{-t} terms mean that the system will converge at long time
- sin and cos terms mean permanent oscillations

Whew!!

Example 1 (cont)

One More Practice Problem

$$Y(s) = \frac{1}{s^2 - 4s + 13}$$

$$s^2 - 4s + 13 = (s-2)^2 + 9$$

$$Y(s) = \frac{1}{(s-2)^2 + 9} = \frac{1}{3}\frac{3}{(s-2)^2 + 3^2}$$

$$y(t) = \frac{1}{3}e^{2t}\sin(3t)$$

Oscillatory, diverges

What if Roots to Denominator Are:

$$\begin{bmatrix} 2+6i \\ 2-6i \\ -1 \\ -3 \\ -2 \end{bmatrix}$$

- $2+6i$: Oscillatory, diverges
- $2-6i$: Oscillatory, diverges
- -1: No oscillations, converges
- -3: No oscillations, converges
- -2: No oscillations, converges

Overall: Oscillatory, diverges